ON MINIMAX LOWER BOUND FOR TIME-VARYING FREQUENCY ESTIMATION OF HARMONIC SIGNAL

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ABSTRACT

Estimation of the instantaneous frequency and its derivatives is considered for a harmonic signal with a time-varying phase and time-invariant amplitude. The asymptotic minimax lower bound is derived for the mean squared error of estimation provided that the phase is an arbitrary m-times piece-wise differentiable function of time. It is shown that this lower bound is different only in a constant factor from the upper bound for the mean squared errors of the local polynomial periodogram with optimal window size.

1. INTRODUCTION

In this paper we consider the problem of estimating the instantaneous frequency (IF) $\Omega(t) = d\varphi(t)/dt$ from complex-valued observations

$$\begin{aligned} y(sT) &= r(sT) + \varepsilon(sT), \\ r(sT) &= A\exp(j\,\varphi(sT))\,,\; s = 1, 2, \ldots N, \end{aligned} \tag{1}$$

where T is a sampling interval and N is the number of observations. The $\{\varepsilon(sT)\}$ are independently and identically distributed Gaussian complex-valued random variables $E\,\varepsilon(sT) = 0$, $E\,\varepsilon(sT)\,\varepsilon^*(sT) = \sigma^2$, where the asterisk means a complex conjugate value. In addition, we assume for the sake of simplicity the independence and equal power of $\mathrm{Re}(\varepsilon(sT))$ and $\mathrm{Im}(\varepsilon(sT))$.

The amplitude A is assumed to be given a priori, while $\varphi(t)$ is an unknown real-valued time-varying phase belonging to the following class of m-times piece-wise differentiable functions

$$\mathcal{F}_m = \left\{ \varphi \,:\, \sup_t \left| \Omega^{(m-1)}(t) \right| \leq L_m \right\}. \tag{2}$$

Here the derivative $\Omega^{(m-1)}(t) = \frac{d^m \varphi(t)}{dt^m}$ is supposed to be a piece-wise continuous function.

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The aim of this study is to find a minimax lower bound for the mean squared estimation error

$$D_k(t) = E\left(\hat{\omega}^{(k-1)}(t) - \Omega^{(k-1)}(t)\right)^2 \tag{3}$$

for the $(k-1)$-th IF derivative $\Omega^{(k-1)}(t)$, $k = 1, \ldots, m-1$, at an arbitrary point t belonging to the observation interval $(t_{\min}, t_{\max})$. Here $\hat{\omega}^{(k-1)}(t)$ is an estimate for $\Omega^{(k-1)}(t)$, $t_{\min} = 0$, $t_{\max} = NT$.

Thus the nonparametric IF and its derivatives $\Omega^{(k-1)}(t)$, $k = 1, \ldots, m-1$, are to be found from the observations (1) which are the nonlinear (exponential) function of the unknown phase. This nonlinearity is a specific feature of the considered problem which makes it quite different from the classical linear setting of nonparametric estimation.

A complex-valued harmonic with the time-varying phase is a key-model of the IF concept as well as the general theory of the time-varying spectrum analysis (e.g. [1] [4] [5]). The Cramer-Rao lower bound of unbiased IF estimation obtained for different parametric models of the signals (e.g., [1] [12]) is a conventional tool for accuracy characterization of IF estimators. However, this lower bound is quite useless for the considered nonparametric setting of the problem when the IF is assumed to be an arbitrary time-varying function and the estimation biasedness is a crucial point. The minimax lower bounds are used as a measure of the best accuracy which could be achieved in nonparametric estimation. In this way the minimax lower bounds for biased estimation play a role similar to that which the usual Cramer-Rao bounds play for unbiased estimation.

This paper is a development of [11] with specification of constants of the minimax lower bounds for estimation of the IF and its first derivative provided that $m = 2$, 3 and 4 in (3). This specification is obtained in [11] only for estimation of the IF and $m = 2$.

The approach developed in this paper is based on the traditional reducing of the nonparametric problem to an auxiliary one with a "worth" parametric family of functions (see, e.g., [6]) and on the information inequality obtained in [10] for parameter estimation in a ball. Note that the related information inequalities have been obtained in [2] [3] [13]. However, the inequality from [10] leads to the more precise minimax lower bounds.

This paper is organized as follows. The main proposition with the lower bounds and discussion are given in section II. The local polynomial periodogram (LPP) as a nonparametric estimator of the IF and its derivatives is presented in section III. It is shown that the derived lower bound is different only in a constant factor from the mean squared error of the LPP with the optimal window size. It proves that the derived lower bound is attainable to within a constant factor, and this bound cannot be principally decreased.

2. MINIMAX LOWER BOUNDS

Proposition 1 ([11]) Let $m > k \geq 1$, and $\hat{\omega}^{(k-1)}(t)$ be an arbitrary estimator for $\Omega^{(k-1)}(t)$ that is a measurable function of observations $\{y(sT)\}$ (1). Then for any fixed $t \in (t_{\min}, t_{\max})$ the following inequality holds:

$$\sup_{\varphi \in \mathcal{F}_m} D_k(t) \geq K_{k,m} \left(L_m^{2k+1} \left(T \frac{\sigma^2}{|A|^2} \right)^{m-k} \right)^{\frac{2}{2m+1}} \tag{4}$$

as $T \to 0$ and $N \to \infty$. Here $K_{k,m}$ is a finite constant depending only on k and m. ■

Comments on Proposition 1.

1. Let the observations be real-valued $y(sT) = A\cos(\varphi(sT)) + \varepsilon(sT)$ with noise variance $E\varepsilon^2 = \sigma^2/2$.

The corresponding minimax lower bound can be given in the following two different forms for the 'slow' and 'fast' varying IF:

(a) Slow IF ($L_m \to 0$)

$$\sup_{\varphi \in \mathcal{F}_m} D_k(t) \geq K_{k,m} \left(L_m^{2k+1} \left(T \frac{2\sigma^2}{|A|^2} \right)^{m-k} \right)^{\frac{2}{2m+1}}, \tag{5}$$

(b) Fast IF ($L_m \nrightarrow 0$, $T/L_m^2 \to 0$)

$$\sup_{\varphi \in \mathcal{F}_m} D_k(t) \geq K_{k,m} \left(L_m^{2k+1} \left(T \frac{\sigma^2}{|A|^2 \sin^2 \varphi(t)} \right)^{m-k} \right)^{\frac{2}{2m+1}}. \tag{6}$$

The comparison shows that the variance σ^2 in (4) is replaced in (5) by $2\sigma^2$ and in (6) by the expression $\sigma^2/\sin^2\varphi(t)$. Thus the minimax lower bound is always larger for the real-valued signal in comparison with the case of the complex-valued observations.

There is also a great difference between the cases of the low and fast IF. This fact deserves discussion.

First, let us stress that the nonparametric estimation asymptotics considered in Proposition 1 assume that the number of observations is increasing in a narrow and narrowing neighbourhood of the time-instant t. It makes it possible to estimate arbitrary fast varying processes. This sort of asymptotics gives the minimax lower bound for the real-valued observation in the form (6). The minimax lower bound depends on the phase $\varphi(t)$ and approaches the infinite value as $\sin^2\varphi(t) \to 0$. The estimates of the frequency are extremely sensitive with respect to the noise when measurements are concentrated in a narrow neighbourhood of the peaks of the real-valued harmonic. The minimax lower bound given by (6) reflects this fact.

Second, the slowly varying IF makes it possible to use observations from a large neighbourhood of the point t covering a large amount of harmonic's periods. In this case the minimax lower bound has the form (5).

2. Calculation of the constant $K_{k,m}$ is based on the function $\kappa(\tau)$ derived in [10], where $\kappa(\tau) : [0,\infty) \to [0,1)$ is a monotonic increasing function, described by its inverse function $\tau = \tau(\kappa)$ as follows:

$$\tau = \frac{\arccos(-\sqrt{\kappa})}{\sqrt{1-\kappa}} - \frac{\pi}{2}, \; \kappa \in [0,1), \tag{7}$$

then [11]

$$K_{k,m} = \left(\max_{w>0} w^{-\frac{2(2k+1)}{2m+1}} \kappa(w) \right) \times \left(\inf_{\mu,\psi} \mu^{2k+1} \left(2 \int_{-1}^{1} \psi^2(u)\, du \right)^{m-k} \right)^{-\frac{2}{2m+1}}. \tag{8}$$

Here inf is calculated with respect to an auxiliary variable $\mu > 0$ and an m-times piecewise differentiable auxiliary function ψ under the following constraints:

$$\psi^{(k)}(0) = 1, \quad \sup_u |\psi^{(m)}(u)| = \mu, \quad \psi(u) = 0 \;\; \forall |u| \geq 1. \tag{9}$$

Note, that the variable μ should be sufficiently large in order to ensure a non-empty set of functions ψ (9). Thus, in order to get the constants $K_{k,m}$, the optimization problems (8)-(9) have to be solved.

3. The following constants have been found as solutions of the above optimization problems:

(a) For estimates of the IF:

$$K_{1,2} = 0.2968, K_{1,3} = 0.1174, K_{1,4} = 0.05277.$$

(b) For estimates of the first derivative of the IF:

$$K_{2,3} = 0.07872, K_{2,4} = 0.08208.$$

3. THE ACCURACY OF THE LOCAL POLYNOMIAL PERIODOGRAM

The discrete-time local polynomial periodogram (LPP) $I_h(\bar{\omega}, t)$ is introduced in the following form [7]-[9]:

$$\begin{aligned} I_h(\bar{\omega}, t) &= |Y_h(\bar{\omega}, t)|^2, \quad (10) \\ Y_h(\bar{\omega}, t) &= \sum_{n=-\infty}^{\infty} \rho_h(nT) y(t + nT) \exp(-j\theta(nT, \bar{\omega})), \\ \theta(u, \bar{\omega}) &= \omega_1 \cdot u + ... + \omega_{m-1} \cdot u^{m-1}/(m-1)!, \\ \bar{\omega} &= (\omega_1, \omega_2, ..., \omega_{m-1})' \in R^{m-1} \end{aligned}$$

where $\rho_h(u) \geq 0$ is a window function and $h > 0$ is a parameter scaling the window size. It is assumed that $\rho_h(nT) = T/h \cdot \rho(nT/h)$, $\int_{-\infty}^{\infty} \rho(u) du = 1$ so that $\sum_{n=-\infty}^{\infty} \rho_h(nT) \to 1$ as $h/T \to \infty$.

The LPP estimator $\hat{\omega}(t, h)$ has been introduced as a solution of the following constrained optimization problem:

$$\hat{\omega}(t, h) = \arg \max_{\bar{\omega} \in Q_\omega} I_h(\bar{\omega}, t). \quad (11)$$

The components of $\hat{\omega}(t, h)$ yield the following estimates: $\hat{\omega}_1(t, h)$ for the frequency $\Omega(t)$ and $\hat{\omega}_{s+1}(t, h)$ for the derivatives $\Omega^{(s)}(t) = d^s\Omega(t)/dt^s$, $s = 1, 2, ...m-2$. It is clear that for $m = 1$ the LPP coincides with the short-time periodogram and gives the estimate of the IF only. Both the convergence result, $\hat{\omega}(t, h) \to \bar{\omega}^0(t)$ as $h/T \to \infty$,

$$\bar{\omega}^0(t) = (\Omega(t), \Omega^{(1)}(t), ..., \Omega^{(m-2)}(t))', \quad (12)$$

and the asymptotic accuracy of the estimator prove that $I_h(\bar{\omega}, t)$ as a function of $\bar{\omega}$ concentrates in the point $\bar{\omega}^0(t)$.

Let us present asymptotic formulas for the covariance and bias of the estimation error

$$\Delta\hat{\omega}(t, h) = \bar{\omega}^0(t) - \hat{\omega}(t, h), \ \ \Delta\hat{\omega}(t, h) \in R^{m-1}. \quad (13)$$

The following notation is used in what follows:

$$S(h) = diag(h, h^2, ..., h^{m-1}), \quad (14)$$

$$\rho = \rho(u), \ \ U = (u, u^2/2, ..., u^{m-1}/(m-1)!)',$$

$$\tilde{\Psi} = \int \rho U U' du - \int \rho U du \int \rho U' du, \quad (15)$$

$$\tilde{\Phi} = \int \rho^2 U U' du - \int \rho U du \int \rho^2 U' du -$$

$$+ \int \rho^2 U du \int \rho U' du + \int \rho^2 du \int \rho U du \int \rho U' du,$$

$$b_s^{(a)} = \frac{1}{s!} \left(\int \rho \left| \tilde{\Psi}^{-1} (U - \int \rho U du) \right| \cdot |u|^s du \right).$$

Proposition 2 Let $\hat{\omega}(t, h)$ be a solution of (11), $T \to 0$, $h \to 0$, $h/T \to \infty$, and $\varphi \in \mathcal{F}_m$, then the covariance of the estimates is as follows

$$cov(S(h)\Delta\hat{\omega}(t, h)) = \frac{\sigma^2 T}{2|A|^2 h} W, \quad (16)$$

$$W = \tilde{\Psi}^{-1} \tilde{\Phi} \tilde{\Psi}^{-1}$$

and the upper bound for the estimation bias is of the form

$$\sup_{\varphi \in \mathcal{F}_m} |E(S(h)\Delta\hat{\omega}(t, h))| \leq h^m b_m^{(a)} L_m. \quad (17)$$

Theorem 2 is a special case of the results given in ([9], Proposition 1).

Comments on Proposition 2.

1. Let us consider the MSE of estimation.

It follows from (16) and (17) that the MSE can be presented in the form

$$\sup_{\varphi \in \mathcal{F}_m} E(\Omega^{(k-1)}(t) - \hat{\omega}_k(t, h))^2 \leq \quad (18)$$

$$\frac{\sigma^2}{2|A|^2 h^{2k+1}} T \cdot W_{kk} + (b_{m,k}^{(a)} h^{(m-k)} \cdot L_m)^2,$$

where W_{kk} is the k-th diagonal item of the matrix W and $b_{m,k}^{(a)}$ is the k-th item of the vector $b_m^{(a)}$.

Minimization of the right-hand side in (18) results in the optimal choice of h and gives

$$h_{0k}^{2m+1} = T \frac{W_{kk}(2k+1)}{4(m-k) b_{m,k}^2} \cdot \frac{\sigma^2}{|A|^2 L_m^2}, \quad k = 1, 2, ..., m-1, \quad (19)$$

and

$$K_{k,m} \leq \frac{\sup_{\varphi \in \mathcal{F}_m} E(\Omega^{(k-1)}(t) - \hat{\omega}_k(t, h))^2}{\left(L_m^{2k+1} \left(T \frac{\sigma^2}{|A|^2} \right)^{m-k} \right)^{\frac{2}{2m+1}}} \leq M_{k,m}, \quad (20)$$

where

$$M_{k,m} = (2m+1) \cdot \left(\frac{W_{kk}}{4(m-k)} \right)^{\frac{2(m-k)}{2m+1}} \cdot \left(\frac{(b_{m,k}^{(a)})^2}{2k+1} \right)^{\frac{2k+1}{2m+1}}, \quad (21)$$

This optimal choice of the scale parameter h determines a trade-off of bias-variance usual for nonparametric estimation. Note that the optimal h_{0k} are different for estimates of the IF and its derivatives.

2. The following values are obtained for the constants $M_{k,m}$:

	Rectang	*Triang*	*Epanech*
$M_{1,2}$	0.5384	0.4972	0.4990
$M_{1,3}$	0.2334	0.2156	0.2144
$M_{1,4}$	0.3205	0.4072	0.4203
$M_{2,3}$	0.7635	0.6416	0.6584
$M_{2,4}$	0.3105	0.2635	0.2703

where the columns correspond to the rectangular, triangular and Epanechnikov window functions, which are respectively of the form: $\rho = 1$, $\rho = 2(1 - 2|u|)$ and $\rho = 3(1 - 4u^2)/2$ for $|u| \leq 1/2$.

It is clear from the table that the window choice influences the accuracy slightly.

Let us compare $K_{k,m}$ to $M_{k,m}$. The values of $K_{1,2}$ and $M_{1,2}$ have quite close values. It proves that for $m = 1$ the *LPP*, i.e. the short-time periodogram, has the accuracy of the IF estimation close to the optimal one. Quite close values have also the constants $K_{1,3}$ and $M_{1,3}$. However, $K_{1,4} \ll M_{1,4}$ as well as $K_{2,m} \ll M_{2,m}$ for $m = 3$ and 4.

3. To the best of our knowledge the LPP is the only estimator of the IF and its derivatives which gives the MSE in the form different from the minimax lower bound in a constant factor depending only on the order of the LPP, the order of the estimated derivative and the window ρ.

In particular, the usual Wigner-Ville distribution enables the estimation bias to be equal to that for the short-time Fourier transform (the LPP with $m = 1$). But the variance of the Wigner-Ville is proportional to $\frac{\sigma^2}{|A|^2}(1+\frac{\sigma^2}{|A|^2})$ instead of $\frac{\sigma^2}{|A|^2}$ for the LPP (e.g. [9] [14]).

4. SUMMARY

The minimax lower bound is derived for the MSE of estimation of the IF and its derivatives provided that the time-varying phase is m-times piece-wise differentiable function of time. The constants of the minimax lower bounds are found for estimation of the IF and its first derivative provided that $m = 2$, 3 and 4.

It is shown that the optimal choice of the window size in the LPP estimator of the IF and its derivatives results in the mean squared errors which are different from the corresponding minimax lower bounds only by a constant factor.

5. REFERENCES

[1] Boashash B. "Estimating and interpreting the instantaneous frequency of a signal– Part1: Fundamentals and Part 2: Algorithms and applications". Proc. IEEE, vol. 80, April, pp. 520–568, 1992.

[2] Borovkov A. A. Mathematical Statistics. Parameter Estimation. Nauka, Moscow, 1984.

[3] Chentsov N. N. Statistical Decision Rules and Optimal Inference. AMS, Providence, 1982.

[4] Cohen L. Time-frequency analysis. Prentice Hall, Englewood Cliffs, N.J., 1995.

[5] Hlawatsch F. and Boudreaux-Bartels G. F. "Linear and quadratic time-frequency signal presentations". IEEE Signal Processing Magazine, vol. 9, February, pp. 21–68, 1992.

[6] Ibragimov I. A. and Khasminskii R. Z. Statistical estimation: asymptotic theory. Springer-Verlag, N.Y., 1981.

[7] Katkovnik V. "Local polynomial periodogram for time-varying frequency estimation". South African Statistical Journal, vol. 29, pp. 169–198, 1995.

[8] Katkovnik V. "Local polynomial approximation of the instantaneous frequency: asymptotic accuracy". Signal processing, vol. 52, pp. 343-356, 1997.

[9] Katkovnik V. "Nonparametric estimation of instantaneous frequency". IEEE Trans. Inf. Theory, vol. 43, 1, pp.183-189, 1997.

[10] Nazin A. V. "On minimax bound for parameter estimation in ball (bias accounting)". In: New Trends in Probab. and Statist., V. Sazonov and T. Shervashidze (Eds.), VSP/Mokslas, pp. 612–616, 1991.

[11] Nazin A. V. and Katkovnik V. "Minimax lower bound for time-varying frequency estimation of harmonic signal". IEEE Trans. on Signal Processing. Submitted, 1997.

[12] S. Peleg and B. Porat, "The Cramer-Rao lower bound for signals with constant amplitude and polynomial phase," IEEE Trans. on Acoustics, Speech, and Signal Processing, vol. 39, pp. 749–752, 1991.

[13] Van Trees H. L. Detection, Estimation and Modulation Theory, Part 1. Wiley, New York, 1968.

[14] Wong K. M. "Estimation of the time-varying frequency of a signal: the Cramer-Rao bound and the application of Wigner distributions". IEEE Trans. on Acoustics, Speech, and Signal Processing, vol. 38, 3, pp. 519–535, 1990.