AN H^∞ APPROACH TO MULTI-SOURCE TRACKING

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ABSTRACT

In this paper, a novel H^∞ approach is proposed for tracking of polarized co-channel sources using an array of tripole antennas. The proposed approach partitions the observation data matrix into two sub-matrices that are used, in conjunction with a new state-space model, to provide an H^∞-type recursive estimation of a *linear combiner*. The linear combiner then provide estimates of the noise and signal subspaces, from which the directions of the incident signals can be estimated and tracked. The proposed technique is also capable of handling the tracking of *appearing / disappearing* sources during the observation interval and, furthermore, can accommodate array modeling uncertainties. The difficult problem of tracking the crossing sources can be successfully handled by using diversely polarized array.

1. INTRODUCTION

Most modern high-resolution direction-of-arrival (DOA) estimation techniques extract information from a subspace of the covariance or data matrix associated with a received signal vector. However, major assumptions associated with these techniques are stationary environment and complete knowledge of the received signal model (i.e., its array manifold and statistical information about the noise signal). To handle a non-stationary environment, which is the case when the sources move, or appear or disappear during the observation, a number of adaptive algorithms for subspace tracking have been developed ([1], [2], and the references therein). These methods provide updates to subspaces at every step of the subspace tracking procedure so that subspace parameter extraction methods such as MUSIC can be used in conjunction with these methods to track the moving sources. However, none of these techniques address the array-uncertainty problem, which still persists, significantly degrading the performance of the subspace-based direction-finding (DF) algorithms and also these methods are suitable only for slowly-varying DOAs. There is, therefore, considerable practical interest in the development of tracking algorithms that can operate in the presence of array uncertainties, rapidly changing DOAs, crossing sources, appearance / disappearance of sources during observation periods, and furthermore, which are less computationally intensive.

The goal of this paper is to outline the use of the H^∞ criterion in the design of robust multi-source tracking algorithm to deal with the problems mentioned in the previous paragraph. The idea of applying H^∞ estimation techniques to these problems is motivated by the fact that the H^∞ estimation is robust and less sensitive to parameter variations, to model uncertainties, and to the lack of statistical information on the noise signal. Furthermore, by using the tripole antenna array we can discriminate multiple sources with respect to their direction and polarization. This property motivated us to use the tripole antenna array to handle tracking of crossing sources.

The paper is organized as follows. In Section-2, the modeling of the received signal for tripole antenna array is given. In Section-3, a new state-space model for the received signal is proposed, which is then placed in the framework of an H^∞ approach. In Section-4, the H^∞ multi-source tracking algorithms is proposed while in Section 5, representative examples are presented. Finally, in Section 6, some concluding remarks are given.

2. MODELING THE ARRAY SIGNAL

The signal-vector $\underline{x}(t)$, received by an array of N tripole antenna operating in the presence of M polarized narrowband sources, can be modeled as follows:

$$\underline{x}(t) = \mathbf{A}\underline{m}(t) + \underline{n}(t), \tag{1}$$

where $\underline{m}(t)$ is the message vector-signal and $\underline{n}(t)$ represents the additive white Gaussian noise. In equation (1), $\mathbf{A}$ is $3N \times M$ complex matrix with its i^{th} column $\underline{a}(\theta_i, \phi_i, \gamma_i, \eta_i)$ defined as $(\mathbf{D}_i\underline{\alpha}_i) \otimes \exp(-j\mathbf{r}^T\underline{k}_i)$, where

$$\begin{aligned}
\underline{k}_i &= \tfrac{2\pi}{\lambda}\left[\cos(\theta_i)\cos(\phi_i),\ \sin(\theta_i)\cos(\phi_i),\ \sin(\phi_i)\right]^T,\\
\mathbf{D}_i &= \tfrac{\lambda}{2\pi}\left[cos^{-1}\phi_i\tfrac{\partial\underline{k}_i}{\partial\theta_i},\ \tfrac{\partial\underline{k}_i}{\partial\phi_i},\ \underline{k}_i\right],\\
\underline{\alpha}_i &= \left[\cos\gamma_i,\ \sin\gamma_i\exp(j\eta_i),\ 0\right]^T,
\end{aligned}$$

$\mathbf{r}$ is the array location matrix and, $\otimes$ denotes Kronecker product. Note that, the source is described by its four parameters azimuth θ_i, elevation ϕ_i, orientation γ_i and ellipticity η. The complex vector $\underline{\alpha}_i$ represents the source polarization, and $\mathbf{D}_i$ is an orthogonal matrix which transforms the source polarization coordinate system to array coordinate system. Note that, a tripole is sensitive to three dimensional components of the electric filed and can be used to exploit both DOA and polarization domain, that is, the tripole antenna array can discriminate multiple sources with respect to their direction and polarization.

It is assumed that the matrix $\mathbf{A}$ is of full rank M, that is, M rows of $\mathbf{A}$ are linearly independent and hence the other rows can be expressed as a linear combination of these M rows. Hereafter, the first M rows are assumed to be linearly independent and the received signal vector $\underline{x}$ is partitioned as follows:

$$\begin{bmatrix} \underline{x}_1 \\ \underline{x}_2 \end{bmatrix} = \begin{bmatrix} \mathbf{A}_1 \\ \mathbf{A}_2 \end{bmatrix} \underline{m} + \begin{bmatrix} \underline{n}_1 \\ \underline{n}_2 \end{bmatrix}. \tag{2}$$

where $\mathbf{A}_1 \in \mathcal{C}^{M\times M}$ and $\mathbf{A}_2 \in \mathcal{C}^{(3N-M)\times M}$. The linear combiner $\mathbf{L} \in \mathcal{C}^{M\times(3N-M)}$ is defined as:

$$\mathbf{L}^H\mathbf{A}_1 = \mathbf{A}_2 \text{ or } \begin{bmatrix}\mathbf{L}^H & -\mathbf{I}_{N-M}\end{bmatrix} \begin{bmatrix} \mathbf{A}_1 \\ \mathbf{A}_2 \end{bmatrix} = \mathbf{T}^H\mathbf{A} = \mathbf{0} \tag{3}$$

where $\mathbf{I}_{(3N-M)}$ and $\mathbf{0}_{(3N-M)\times M}$ are the identity and the null matrices, respectively, and $\mathbf{T} \in \mathcal{C}^{3N\times(3N-M)}$. Equation (3) implies that the steering vectors $\underline{a}(\underline{\psi}_i)$ are orthogonal to the columns of $\mathbf{T}$, where $\underline{\psi}_i = [\theta_i, \phi_i, \gamma_i, \eta_i]^T$. This means that the subspace spanned by the columns of the matrix $\mathbf{T}$, span{ $\mathbf{T}$ }, is included in span{ $\mathbf{E}_n$ }, where $\mathbf{E}_n$ is noise subspace, i.e, the eigenvectors associated with the smallest eigenvalues of the data covariance matrix $\mathbf{R}_{xx}$. Now, since $\mathbf{T}$ contains the block $\mathbf{I}_{3N-M}$, its $3N - M$ columns are linearly independent, therefore,

$$\text{span}\{\mathbf{T}\} = \text{span}\{\mathbf{E}_n\}. \tag{4}$$

It follows that the linear combiner defines the noise subspace. Note that, in contrast to the basis defined by $\mathbf{E}_n$, the basis defined by the columns of $\mathbf{T}$ is not orthonormal. However, the result of applying Householder transforms to $\mathbf{T}$ will enable us to find an orthonormal basis for $\mathbf{T}$ and an orthonormal basis for its orthonormal complement, which is the signal subspace, $\mathbf{E}_s$. Furthermore, it is shown in [3] that the following MUSIC-like function can be defined to estimate the DOAs

$$F_{LC} = \lambda_{\text{min}}\left(\begin{bmatrix} \underline{a}_x^H(\underline{\psi}) \\ \underline{a}_y^H(\underline{\psi}) \end{bmatrix} \mathbf{T}\mathbf{T}^H \begin{bmatrix}\underline{a}_x(\underline{\psi}) & \underline{a}_y(\underline{\psi})\end{bmatrix}\right), \tag{5}$$

where $\underline{a}_x(\underline{\psi})$ and $\underline{a}_y(\underline{\psi})$ are two steering vectors corresponding to distinct polarizations and $\lambda_{\text{min}}(\cdot)$ is the minimum eigenvalue of $(\cdot)$. In this investigation, equation-(2) will be used as the starting point to arrive an alternative modeling and then to propose a new approach for improving the performance of multi-source tracking algorithm operating in the presence of array uncertainties.

3. THE H^∞ APPROACH

The H^∞ estimation methods can be seen as a powerful and robust solution to handle parameter variations, array uncertainties and noise effects with limited statistical information. The idea is to come up with estimators that minimize (or in the suboptimal case, bound) the maximum energy gain from the disturbances to the estimation errors [4]. This will guarantee that if the disturbances are small (in energy) then the estimation errors will be as small as possible (in energy), no matter what the disturbances are. The robustness of the H^∞ estimators follows from this fact.

To apply H^∞ estimation techniques to array signal processing, a new state-space model for the received signal of a general array of sensors is developed. Using equations (2) and (3), the $\underline{x}_2$ portion of the received signal can be written as,

$$\underline{x}_2 = \mathbf{L}^H\underline{x}_1 + (\underline{n}_2 - \mathbf{L}^H\underline{n}_1). \tag{6}$$

It can be shown that the $\underline{x}_2$ portion of the received signal vector at a particular time j obeys the following state-space model:

$$\begin{cases} \mathbf{L}_{j+1} &= \mathbf{L}_j + \Delta\mathbf{L}_j, \quad j \in [1, K] \\ \underline{x}_{2,j} &= \mathbf{L}_j^H\underline{x}_{1,j} + \underline{v}_j \end{cases} \tag{7}$$

where $\underline{v}_j = (\underline{n}_{2,j} - \mathbf{L}_j^H\underline{n}_{1,j})$ may include model uncertainties and $\Delta\mathbf{L}_j$ represents the time variation in the state matrix. As it is unknown, we shall consider it as a disturbance. In state-space terminology, the vector $\underline{v}_j$ is the *measurement noise* while the matrix $\mathbf{L}_j$ is the state matrix.

By using the above state space model, the objective is to estimate the unknown state matrix. Let $\hat{\mathbf{L}}_j = \mathcal{F}(\underline{x}_{2,1}, \ldots, \underline{x}_{2,j-1})$ denote the estimate of $\mathbf{L}_j$ given the observations $\{\underline{x}_{2,i}\}$ and $\{\underline{x}_{1,i}\}$ from time 1 up to and including time $j-1$. Let $\mathbf{\Pi}_0$ be a given positive-definite matrix and choose any initial estimate for $\mathbf{L}_0$, which we shall denote by $\hat{\mathbf{L}}_0$. Define the weighted disturbances $\underline{\tilde{L}}_0$ and $\underline{\tilde{\Delta}}_i$ as well as the estimation error $\underline{e}_i$ as follows

$$\begin{cases} \underline{\tilde{L}}_0 &\equiv \mathbf{\Pi}_0^{1/2}\underline{\sigma}_{\text{max}}\left(\mathbf{L}_0 - \hat{\mathbf{L}}_0\right), \\ \underline{\tilde{\Delta}}_i &\equiv \mathbf{\Upsilon}_0^{1/2}\underline{\sigma}_{\text{max}}\left(\mathbf{\Delta L}_i\right), \\ \underline{e}_j^H &\equiv \underline{x}_{1,j}^H\mathbf{L}_j - \underline{x}_{1,j}^H\hat{\mathbf{L}}_{j|j-1}, \end{cases} \tag{8}$$

where $\mathbf{\Upsilon}_0$ is a positive definite matrix that reflects a priori knowledge of how rapidly the state matrix $\mathbf{L}_j$ varies with time and $\underline{\sigma}_{\text{max}}(\cdot)$ is the maximum singular vector of $(\cdot)$. For every choice of estimator $\mathcal{F}(\cdot)$ we will have a transfer operator from disturbances

$$\left\{\underline{\tilde{L}}_0 +, \{\underline{v}_i\}_{i=0}^{j-1}, \{\underline{\tilde{\Delta}}_i\}_{i=0}^{j-1}\right\}$$

to the state prediction error

$$\{\underline{e}_1, \underline{e}_2, \ldots, \underline{e}_j\},$$

which we shall denote by $\mathcal{T}_j(\mathcal{F})$. In the H^∞ framework, robustness is ensured by minimizing the maximum energy gain from the disturbances to the estimation errors. This leads to the following problem:

Problem (The Time-Varying Problem) Find an H^∞ optimal estimation strategy $\hat{\mathbf{L}}_j = \mathcal{F}(\underline{x}_{2,1},\ldots,\underline{x}_{2,j-1})$ that minimizes $\|\mathcal{T}_j(\mathcal{F})\|_\infty$, and obtain the resulting

$$\begin{aligned}\epsilon_g^2 &= \inf_{\mathcal{F}} \|\mathcal{T}_j(\mathcal{F})\|_\infty^2 \qquad (9)\\ &= \inf_{\mathcal{F}} \sup_{\underline{\tilde{L}}_0,\underline{\tilde{\Delta}}_i,\underline{v}} \frac{\sum_{i=1}^{j} \underline{e}_i^H \underline{e}_i}{\|\underline{\tilde{L}}_0\|_2^2 + \sum_{i=1}^{j-1} \underline{v}_i^H \underline{v}_i + \sum_{i=1}^{j-1} \|\underline{\tilde{\Delta}}_i\|_2^2}\end{aligned}$$

□

We shall assume, without loss of generality, that $\mathbf{\Pi}_0$ and $\mathbf{\Upsilon}_0$ have the special form $\mathbf{\Pi}_0 = \mu\mathbf{I}$ and $\mathbf{\Upsilon}_0 = \rho\mathbf{I}$, where μ and ρ are positive constants. Note that for a filter that varies slowly with time, ρ will typically be very small (in the simulation examples bellow, we choose $\rho = 0.004$ and $\mu = 0.9$).

Solution (The Time-Varying Algorithm) The solution to the above problem is given as follows [4]: Consider model (7), and choose

$$\epsilon_g^2 \le \sup_j \left[\rho + \frac{1}{\underline{x}_{1,j}^H \underline{x}_{1,j}}\right] \underline{x}_{1,j+1}^H \underline{x}_{1,j+1} \quad j = 0,\ldots,i. \qquad (10)$$

The central H^∞ optimal estimator is then given by

$$\hat{\mathbf{L}}_{j+1} = \hat{\mathbf{L}}_j + \frac{\tilde{\mathbf{P}}_j \underline{x}_{1,j}}{1 + \underline{x}_{1,j}^H \tilde{\mathbf{P}}_j \underline{x}_{1,j}} \left(\underline{x}_{2,j}^H - \underline{x}_{1,j}^H \hat{\mathbf{L}}_j\right), \quad \hat{\mathbf{L}}_0 \qquad (11)$$

where

$$\tilde{\mathbf{P}}_j^{-1} = \mathbf{P}_j^{-1} - \epsilon_g^{-2} \underline{x}_{1,j} \underline{x}_{1,j}^H \qquad (12)$$

and $\mathbf{P}_j$ satisfies the recursion

$$\mathbf{P}_{j+1} = \left[\mathbf{P}_j^{-1} + (1 - \epsilon_g^{-2}) \underline{x}_{1,j} \underline{x}_{1,j}^H\right]^{-1} + \mathbf{\Upsilon}_0 \qquad (13)$$

initialized with $\mathbf{P}_0 = \mathbf{\Pi}_0$. □

4. THE PROPOSED ALGORITHM: $\mathbf{H^\infty MSTA}$

Increase in the Number of Sources: Let $\mathbf{A}$ be the matrix whose columns consist of the estimated steering vectors generated using the $\underline{\psi}$ found in the previous iteration. When the number of sources increases from M to $M+1$, the data snapshot $\underline{x}(t+1)$ will contain a component not in the span of the matrix $\mathbf{A}$. This can be detected by taking the orthogonal projection of the data snapshot $\underline{x}(t+1)$ onto the subspace spanned by the estimated steering matrix, i.e,

$$\underline{x}'(t+1) \triangleq \mathbf{A}\left(\mathbf{A}^H\mathbf{A}\right)^{-1}\mathbf{A}^H \frac{\underline{x}(t+1)}{\|\underline{x}(t+1)\|} \qquad (14)$$

and plotting $1 - \|\underline{x}'(t+1)\|$ versus the iteration number. There will be a peak in this plot when an increase in the number of sources occurs. At this time, the algorithm is reinitialized and the number of sources is set to $M = M+1$.

Decrease in the Number of Sources: The method for detecting the disappearance of sources is to estimate the based band signal $\underline{m}(t)$ using the least square technique, i.e,

$$\underline{\hat{m}}(t) = \left(\mathbf{A}^H\mathbf{A}\right)^{-1}\mathbf{A}^H\underline{x}(t). \qquad (15)$$

When the sources disappear, the corresponding elements will be very small, given by $\left(\mathbf{A}^H\mathbf{A}\right)^{-1}\mathbf{A}^H\underline{n}(t)$, from which the source disappearance is detected. The algorithm is then reinitialized and the number of sources set to $M = M-1$.

The proposed algorithm, which is called the $\mathbf{H^\infty MSTA}$ (H^∞ **Multi-Source Tracking Algorithm**), can be presented in step format, as follows:

<u>$\mathbf{H^\infty MSTA}$ **Algorithm**</u>

(*i*) Initialization: estimate the number of sources M and the initial state $\mathbf{L}_0$ using the first few snapshots.

(*ii*) Estimate the linear combiner $\mathbf{L}_j$ using the proposed H^∞ estimation technique, given in equation (11).

(*iii*) Use the estimated $\mathbf{L}_j$ to form the noise subspace $\mathbf{E}_{n,j}$ and estimate the DOAs at time j (equations (3) and (5) respectively).

(*iv*) Determine the change in number of sources at time $j+1$ (received signal $\underline{x}_{j+1}$).

IF $1 - \|\underline{x}'(t+1)\| >$ threshold (based on equation (14))
THEN $M = M+1$ and re-estimate the initial state matrix $\mathbf{L}_0$.

ELSEIF elements of $\underline{\hat{m}}(t) <$ threshold (based on equation (15))
THEN $M = M-1$ and re-estimate the initial state matrix $\mathbf{L}_0$.

ELSE No change.

(*v*) Go to step (ii) until $j = K$ (observation interval).

5. REPRESENTATIVE EXAMPLES

It is well known that imprecise knowledge of array characteristics can seriously degrade the performance of a DF algorithm. Therefore in this section an attempt is made to quantify this degradation by showing a representative example with respect to errors in the tracking performance resulting from perturbations in the array manifold. The simulation environment contain a uniform linear array of 10 tripole sensors operating in the presence of a two moving polarized sources ($\gamma_1 = 45^o, \eta_1 = 0, \gamma_2 = 0, \eta_2 = 45^o$) of equal-power with an angular velocity of -0.1^o and -0.2^o per snapshot. The signal-to-noise power ratio is taken to be equal to 20dB, and the array propagation errors are assumed to be random, uncorrelated with the noise, with zero-mean and second-order moments given by

$$\begin{aligned}\mathcal{E}\left\{\underline{\tilde{a}}(\theta_i)\cdot\underline{\tilde{a}}^H(\theta_j)\right\} &= \sigma_a^2\mathbf{I}\delta_{i,j}\\ \mathcal{E}\left\{\underline{\tilde{a}}(\theta_i)\cdot\underline{\tilde{a}}^T(\theta_j)\right\} &= \mathbf{0}.\end{aligned}\qquad(16)$$

The array response is perturbed according to Equation (16) for each snapshot, with $\sigma_a = 0.2$. Figures (1) and (2) shows the result of the Yang's PASTd [2] and proposed algorithm for this signal environment, where the solid line represents true directions and 'o' indicates the estimated directions. It is clear from the results that the $\mathbf{H^\infty MSTA}$ correctly tracks the sources, while the PASTd algorithm does so, but with less accuracy. Thus due to the H^∞ formulation, $\mathbf{H^\infty MSTA}$ is robust with respect to array calibration error.

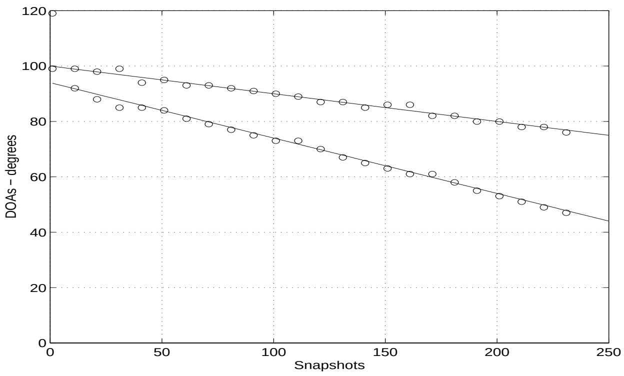

Figure 1: PASTd Algorithm

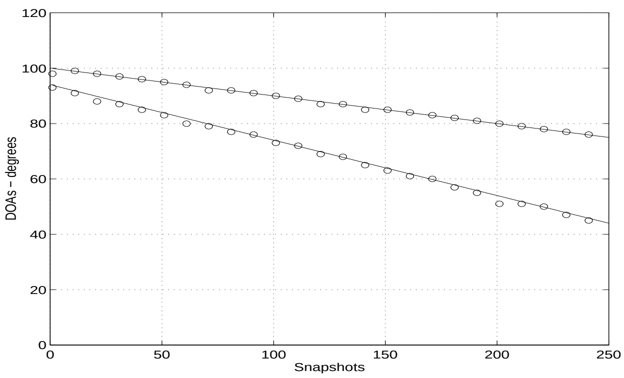

Figure 2: $\mathbf{H^\infty MSTA}$

Figure (3) shows the simulation result which demonstrates the source crossing signal environment; two moving sources with an angular velocity of 0.1^o and -0.1^o per snapshot. It is clear from the figure that the $\mathbf{H^\infty MSTA}$ successfully handle the problem of sources crossing.

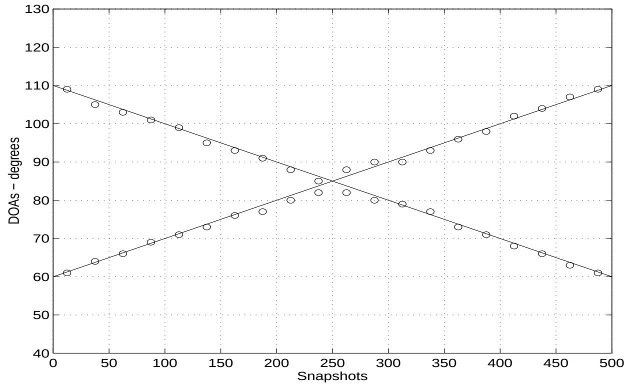

Figure 3: $\mathbf{H^\infty MSTA}$

6. CONCLUSION

In this paper, we proposed a new H^∞ approach to multi-source tracking problem. The algorithm is capable of handling rapidly changing DOAs, crossing sources, appearing / disappearing sources during the observation interval and, furthermore, can accommodate array modeling uncertainties. In addition, the algorithm is recursive in nature therefore less computationally intensive. Finally, the proposed algorithm is compared with the well known PASTd algorithm of [2]. From the simulation results, it has been shown that the proposed technique out-performed the PASTd when the array uncertainty exists.

7. REFERENCES

[1] R.D. DeGroat, E.M Dowling, H. Ye, and D.A. Linebarger, "Spherical Subspace Tracking for Efficient, High Performance Adaptive Signal Processing Applications", *Signal Processing* Vol. 50, pp. 101-121, 1996.

[2] B. Yang, "Projection Approximation Subspace Tracking", *IEEE Trans. on Signal Processing*, Vol. 43, No. 1, pp 95-107, January 1995.

[3] J. Munier and G.Y Delisle, "Spatial Analysis Using new Properties of the Cross-Spectral Matrix", *IEEE Trans. on Signal Processing*, Vol. 39, No. 3, pp 746-749, March 1991.

[4] B. Hassibi and T. Kailath, "H^∞ Adaptive Filtering", *In Proc. of ICASSP'95*, pp. 949-952.