DENOISING BY EXTRACTING FRACTIONAL ORDER SINGULARITIES

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ABSTRACT

In this paper we will introduce a method of isolating and extracting certain class of local singular behaviours of signals/images which in turn will lead to a method of pointwise noise estimation and suppression. The underlying motivation is to decompose functions directly in terms of components which would naturally represent different orders of regular or singular behaviours defined by the local Hölder exponents. We have shown that such a decomposition can lead to a factorization of the spectrum of the singular portion of the signal in terms of the spectrum of the original signal and that of a denoising filter.

1. CLASSICAL DENOISING METHODS

These methods can be broadly divided into two main categories: those based on statistics and the ones based on the extraction of the regularity. In the former category the early tools were based on linear filtering which were rapidly replaced by non-linear tools mostly based on order statistics (L-estimators [8]). The most well known filter in this category is the median filter originally introduced by Tuckey in time series analysis [10]. Since then, many different versions of this class of filters have been introduced [8].

In the second category one can identify two trends: one is based on the local regularity of a function and the other on the notion of the global regularity. The first group is based on the assumption that if the sampling period is small enough then the Taylor series expansion of the signal can be locally truncated to the kth order, in which case one can fit locally a kth order polynomial using $n > k$ data points. The problem will then reduce to that of optimizing a cost function so that k local parameters can be determined using n observations. Therefore, in this method every observation is combined with its immediate neighbours in order to identify the analytic nature of the underlying signal in the neighbourhood of each sampling point. The method is, thus, imposing local kth order regularity.

In the second trend regularity of a given signal is considered globally. The most simple way of defining global regularity is in the Fourier domain using the results based on Riemann lemma [7], which suggests that the Fourier series expansion of the regular and the irregular portions of a function have different convergence rates. A whole set of tools have been developed for extracting irregularity on this basis. As for the irregularities the common assumption is non-analyticity.

The advantage of this approach is simplicity, since one would only need to design a filter according to some optimality criterion and then obtain the results by a simple multiplication in the Fourier domain. However, whatever the optimality criterion, their performance is usually limited by the choice of some parameter depending on the noise statistics (the cut-off frequency for example).

2. MORE RECENT METHODS

All above mentioned methods would considerably loose their performance when the noise is more prominent or has a higher power than the underlying signal, and therefore highly depend on the noise variance and/or its estimation. Recent improved methods are based on the extraction of local regularity using Lipschitz or Hölder exponents:

Definition 1 *A function $f(t)$ is said to belong to a class of functions that are Lipschitz of order α (and is denoted by $f \in Lip_\alpha$), $\alpha \in]0,1]$, if for any t_0 and an arbitrarily small ϵ, there exists a positive finite constant c such that:*

$$| f(t_0) - f(t_0+\epsilon) | \leq c \mid \epsilon \mid^\alpha \quad (1)$$

It follows that for $\alpha \in]0,1[$ the function is singular at t_0 and for $\alpha = 1$ it is differentiable at t_0. Higher order regularities (or lower order singularities) can be defined by $k < \alpha \leq k+1$, where $k \neq 0$ is an integer.

Extensive literature is available dealing with theoretical aspects of Hölder exponents. Recently, algorithms have also been proposed in signal/image processing which use these results together with multi-scale analysis based on wavelet transforms for extracting local isolated singularities, and denoising [1][2][3][4][6].

Most of these methods propose a three stage approach: decomposition, characterization of singularities followed by some sort of thresholding and reconstruction. Both decomposition and reconstruction are based on wavelet transform (of possibly different types). The major difference between

The fist author was at INRIA when this research was carried out.

these algorithms is probably at the second stage, namely in the way they characterize singularities whether for eliminating or preserving.

Our motivation is to combine these three stages. An obvious approach would be to decompose a given signal directly in terms of components that would naturally separate different orders of singularities, in which case the extraction of singularities of particular order could be easily achieved by removing the corresponding terms. The following section will sketch a background for this.

3. EXTRACTING INTEGER AND FRACTIONAL ORDER SINGULARITIES

We will first consider the problem for integer order singularities. This will give us an insight to the problem at fractional orders. From the calculus of complex functions we know that:

Definition 2 *A function $f(z)$ is analytic in a simply connected region Υ if:*

$$\oint_{\Upsilon} f(z)dz = 0 \tag{2}$$

where the integral is taken around any simple closed Jordan curve inside the region Υ.

The extension to multiply connected regions is straightforward and well known. Note also that analyticity in the complex plane is more restrictive than the one along the real axis, which is simply defined by the convergence of the Taylor series (ie. when $f \in \mathcal{C}^{\infty}$). There is, however, an equivalent expansion in the complex plane if we assume that f is singleton (ie. no branch cuts). This expansion is given by Laurent series:

$$f(z) = \sum_{r=-\infty}^{\infty} a_r(z-z_0)^r \tag{3}$$

$$\text{where} \quad a_r = \frac{1}{2\pi i}\oint_C \frac{f(z)dz}{(z-z_0)^{1+r}} \tag{4}$$

where z_0 is any point inside the contour of integration C.

For the functions to be considered hereafter, we will assume that, for every point z_0, there exists an annulus $A_0 <| z - z_0 |< A_1$ in the complex plane inside which the series is uniformly convergent.

Definition 3 *When $A_0 = 0$ and f is analytic everywhere on a disc $0 <| z - z_0 |< A_1$, except at $z = z_0$, then z_0 is said to be an isolated singularity of f and $\Upsilon - \{z_0\}$ is said to be a deleted region of z_0.*

We will see below that isolated singularities can be easily removed. This is done by simply removing the so called principal part of the Laurent series:

$$\begin{aligned} f_p(z) &= \sum_{r=-\infty}^{-1} a_r(z-z_0)^r \\ &= \sum_{r=1}^{\infty} a_{-r}(z-z_0)^{-r} \end{aligned} \tag{5}$$

Theorem 1 *Let $f(z)$ be a function defined on $\mathbb{C}$ with a set of (possibly denumerable) isolated integer order singularities. Let also the non-singular portion of f be an $\ell_2(\mathbb{C})$ band-limited function. Then the Fourier transform of the principal part of its Laurent series expansion at any singular point z_0 satisfies:*

$$\hat{f}_p(u)|_{z=z_0} = \hat{f}(u)\left(\frac{1}{iu-1}\right) \tag{6}$$

For proof see [9].

The immediate consequence of the above theorem is that integer order isolated singularities of a function may be simply removed by multiplying its spectrum by $\frac{1}{iu-1}$. However, this result is of no practical interest since the order of local regularity of functions is measured by α (as defined earlier), which is a continuous parameter in $\mathbb{R}$. Using the above result one can not isolate singularities of non-integer order that lie in the interval $]-1,0]$.

To extend the results to fractional singularities, we use the Riemann series which is defined using fractional derivatives:

Definition 4 *(Riemann series) The Riemann series of a function f on $\mathbb{C}$ is defined by:*

$$f(z) = \sum_{r=-\infty}^{\infty} a_r(z-z_0)^{r+\alpha} \tag{7}$$

$$a_r = \frac{\mathcal{D}^{r+\alpha}_{z-b} f(z)}{\Gamma(1+r+\alpha)} \tag{8}$$

where, $\mathcal{D}^{r+\alpha}_{z-b}$ denotes the fractional derivative operator of order $r+\alpha$ with respect to $z-b$, α is an arbitrary complex number, and Γ is Euler's gamma function.

Definition 5 *(fractional derivative) The fractional derivative of f is given by Cauchy's integral formula:*

$$\mathcal{D}^{r+\alpha}_{z-b} f(z) = \frac{\Gamma(1+r+\alpha)}{2\pi i}\oint_{\Upsilon} \frac{f(z)dz}{(z-z_0)^{1+r+\alpha}} \tag{9}$$

Theorem 2 *Let $f(z)$ be a function defined on $\mathbb{C}$ with a set of (possibly denumerable) isolated non-integer or integer order singularities. Let also the non-singular portion of f be an $\ell_2(\mathbb{C})$ band-limited function. Then the Fourier transform of the principal part of its Riemann series expansion at any singular point z_0 satisfies:*

$$\hat{f}_p(u)|_{z=z_0} = \lim_{\alpha\to -1^+} \hat{f}(u)\mathcal{K}(u)$$
$$\text{where } \mathcal{K}(u) = \left(\frac{u^{-\alpha}}{1+u^2}\left(u\ exp(-i\theta) - exp\left(i\frac{\pi}{2} - i\theta\right)\right)\right) \tag{10}$$

with $\theta = \frac{\pi}{2}(1+\alpha)\mathrm{sgn(u)}$ and sgn the signum function.

See [9] for proof.

The above theorem is interesting since it implies that non-integer singularities below a given order $r+\alpha$ can be removed using the filter $\mathcal{K}(u)$. This is equivalent to expanding a function directly in terms of Hölder exponents and removing the terms that correspond to exponents below a given value.

4. IMPLEMENTATION AND RESULTS

To implement the filter in (10), we notice that:

$$\lim_{\alpha \to -1^+} \mathcal{K}(u) = \frac{u}{1+u^2}(u-i) \qquad (11)$$

which corresponds to the spectrum of a one-sided exponential and hence is separable in multi-dimensions. It, therefore, follows that at the limit the following filter will extract the regular portion of a given signal:

$$1 - \mathcal{K}(u) = \frac{1-iu}{1+u^2} \qquad (12)$$

To apply the results for denoising, we recall that the Gaussian noise is a nowhere differentiable continuous function [5], whose Hölder exponents are uniformly negative [6], whereas regular functions are uniformly Lipschitz positive. This implies that if the observation contains Gaussian noise, then the noise removal can be simply achieved by applying the above filter.

Figure 1 shows some simulation results for a noisy observation of signal to noise ratio SNR $\simeq$ 7dB which has increased to 17dB after suppression. In Figure 2 we have a simulation for a worse case where the noisy observation was at SNR $\simeq$ 1dB and increased to almost 13dB after suppression.

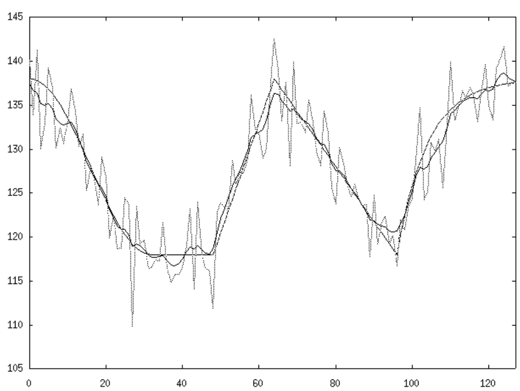

Figure 1: Test signal, noisy observation *(SNR $\simeq 7dB$)* and the restored signal *(SNR $\simeq 17dB$)* superimposed

Many simulations for 2D case were also performed. In the first set, shown below, we have SNR $\simeq$ 2.6dB for the corrupted image and SNR $\simeq$ 11.5dB for the restored image. In the second set the noise power in the corrupted image is higher than the signal power: SNR $\simeq -2$dB. The restored image has SNR $\simeq$ 8.5dB.

We have compared our method with a median filter. Several experiments were performed by different kernel sizes for the median filter. The results shown below seem to be the best that can be achieved by a median filter. In the third set we have used a test image containing fine details. Results of denoising show that our filter performs better.

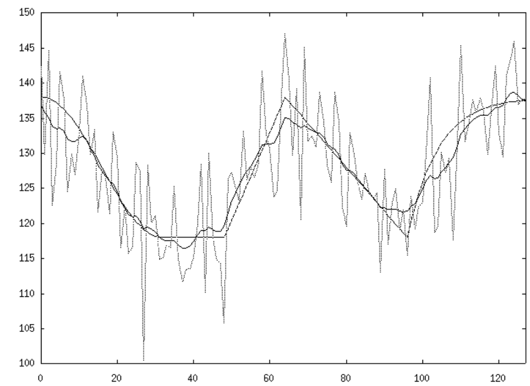

Figure 2: Test signal, noisy observation *(SNR $\simeq 1dB$)* and the restored signal *(SNR $\simeq 13dB$)* superimposed

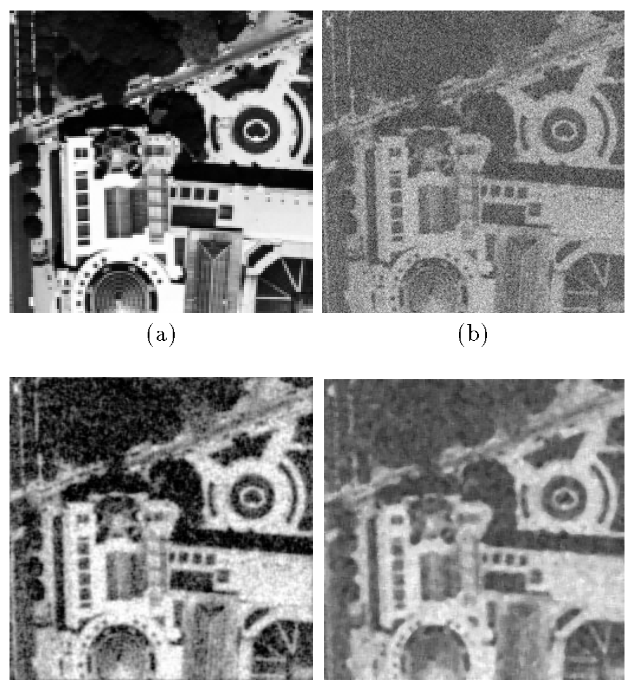

Figure 3: (a) Test image, (b) noisy observation *(SNR $\simeq$ 2.6dB)*, (c) our method *(SNR $\simeq 11.5dB$)*, (d) median filter *(SNR $\simeq 7.3dB$)*

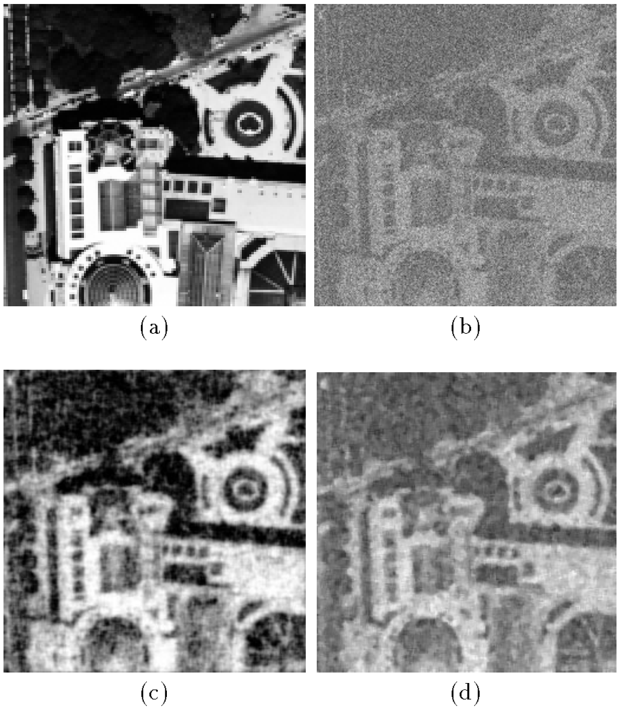

Figure 4: (a) Test image, (b) noisy observation *($SNR \simeq -2dB$)*, (c) our method *($SNR \simeq 8.5dB$)*, (d) median filter *($SNR \simeq 6dB$)*

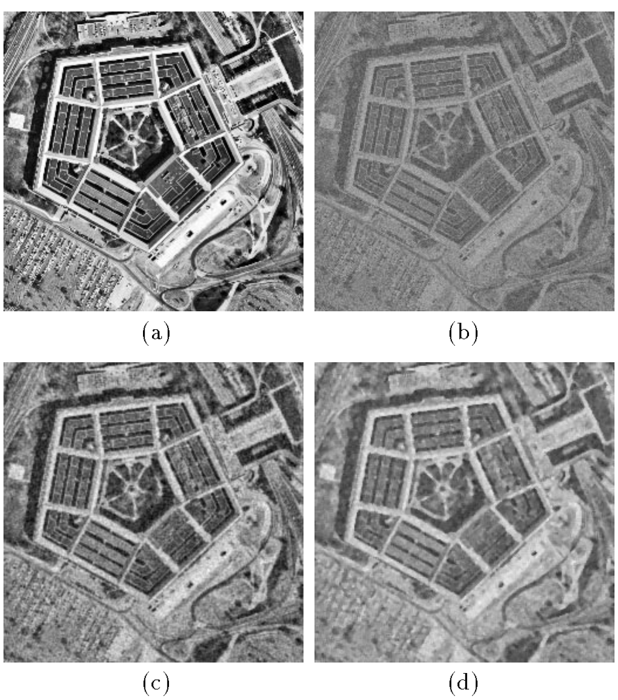

Figure 5: (a) Test image, (b) noisy observation *($SNR \simeq 6dB$)* (c) our method *($SNR \simeq 9dB$)*, (d) median filter followed by histogram equalization *($SNR \simeq 6.4dB$)*

5. CONCLUSION

In this paper we have shown that removing singularities can be achieved by simple filtering. The filter is obtained by developing the observed signal in Riemann series expansion and then truncating the series to the regular portion. The spectrum of the truncated version will then factorize into the spectrum of the original function and the denoising filter. The filter happens to have an anti-causal exponential impulse response which is separable in higher dimensions.

The advantage of this approach to other decompositions such as wavelet transforms is that in Riemann series the components are naturally characterizing different orders of Hölder exponents (ie. different orders of singular/regular behaviours). On the other hand, since they are organized in ascending order of regularity, a simple truncation can separate the singularities.

However, notice that, in order to obtain a closed form solution to our filter, we had to make two main assumptions: (1) that the singularities were isolated, (2) that the regular portion was analytic. The latter assumption may smooth some of the singularities that are needed to be preserved. Nevertheless, the results of simulations for noise suppression demonstrate the superiority of our filter compared to the classical median filter. The performance of the filter is particularly standing out in very low SNR or when the noise power is higher than that of the signal.

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