THE CONNECTION BETWEEN CONTINUOUS AND DISCRETE LATTICE FILTERS

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ABSTRACT

The importance of lattice structures in connection with filtering and prediction has been known for decades. The demand for faster processing has led to steadily increasing sampling rates, and as a result the behavior of the discrete filters as the sampling period tends to zero has become an important theoretical and practical issue. One way of solving the numerical problems that arise in the usual filter structures when the sampling period becomes small compared with the dynamics of the underlying physical processes is to resort to δ operators instead of delay operators. Although the interrelations between the continuous and discrete lattice structures have been rarely studied, it is known that the δ lattice naturally leads to a continuous form as the sampling rate increases. This paper addresses this point and establishes the rate of convergence of the discrete lattice filter to the continuous filter as a function of the sampling period or of the filter order.

1. INTRODUCTION

The importance of discrete and continuous lattice structures in connection with digital signal processing in general, and filtering, autoregressive modeling, and prediction in particular, has been well known for decades [4, 6].

The digital signals processed by the discrete lattice filters, as in the case of other digital filters, are often obtained by sampling some continuous-time signal. However, the asymptotic behavior of the digital system at high sampling rates is rarely studied.

This is an important theoretical and practical issue, because the demand for faster processing has led to steadily increasing sampling rates, and most traditional numerical algorithms become increasingly ill-conditioned as the sampling rate increases.

One way of solving the numerical problems that arise in the usual processing structures when the sampling period tends to zero is to resort to δ operators instead of delay operators [1–3, 10]. Although the interrelations between the continuous and discrete lattice structures have been rarely studied, it is known [10] that the δ lattice naturally leads to a continuous form as the sampling rate increases.

We follow the notation and terminology used in [10]. The bar in $\bar{x}(k)$ or $\bar{x}_k$ is used to avoid an explicit reference to the sampling period Δ: $\bar{x}(k)$, or $\bar{x}_k$, are shorthands for $x(k\Delta)$. The duration of the observation interval, $[t-T, t]$, is T (below we will take $T = 1$ for simplicity). The observed signal y is the sum of a unknown signal s in white noise v. The prediction / smoothing problem consists in predicting s at a specific time t, given y for all $t \in [t-T, t]$. It is assumed that s is zero-mean wide sense stationary, and that s and v are uncorrelated. The covariance of s is known, and denoted by $W(\tau)$ (in the continuous case) or $\bar{W}_i$ (discrete case). The estimate $\hat{s}$ of s, given the observed signal, is

$$\hat{s}(t) = -\int_{t-T}^{t} A(T, t-\sigma) y(\sigma)\, d\sigma, \tag{1}$$

or, in the discrete case,

$$\hat{\bar{s}}(k+1) = -\sum_{i=1}^{N} \bar{K}^i \, \bar{e}_b(i-1, k),$$

where $\bar{K}^i$ are the reflection coefficients, the $\bar{e}_b$ are the backward residuals, and $A(T, \tau)$ or simply $A(\tau)$ defines the continuous time prediction filter (see [10]). The Levinson predictor is optimum in the mean square sense and works in terms of a weighted sum of past values of the measurement signal y,

$$\hat{\bar{s}}(k+1) = -\sum_{i=1}^{N} \bar{a}_i^N \bar{y}(k-i+1),$$

and the weights $\bar{a}^N$ can be obtained by solving the Yule-Walker equations. They are also called the reflection coefficients, since $\bar{a}_N^N$ corresponds to the coefficient $\bar{K}^N$ in the lattice filter.

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Weller *et al.* [10] define the Δ scaled variables

$$\bar{A}(N,i) := \frac{\bar{a}_i^N}{\Delta}, \quad 1 \le i \le N,$$

and $\bar{A}_{N+1}^{N+1} = \bar{K}^{N+1}/\Delta$. In terms of these variables one has

$$\hat{s}(k+1) = -\sum_{i=1}^{N} \bar{A}(N,i)\bar{y}(k-i+1)\Delta,$$

which should be compared with (1).

It was shown by Weller *et al.* [10, Theorem 5.1] that, under certain conditions and with $N\Delta = T$,

$$\lim_{N\to\infty} \|A(\tau) - \bar{A}^{\Delta}(\tau)\|_\infty = 0, \tag{2}$$

for $\tau = 0, \Delta, \ldots, (N-1)\Delta$. Here, $\bar{A}^\Delta$ is a continuous-time piecewise constant function constructed from the $\bar{A}(N,i)$ in the following way:

$$\bar{A}^{\Delta}(\tau) = \begin{cases} \bar{A}(N,i), & \tau \in [(i-1)\Delta, i\Delta], 0 < i < N \\ \bar{A}(N,N), & \tau \in [(N-1)\Delta, T]. \end{cases}$$

This leaves open one important question: how rapidly does $\bar{A}^\Delta$ converge to A as $\Delta \to 0$?

This paper addresses this point and establishes the rate of convergence of the discrete lattice filter to the continuous filter, as a function of the sampling period Δ or of the filter order N. Our results show that the convergence of (2) can be at least $O(1/N)$ in the L_∞ norm.

Weller *et al.* constructed a piecewise constant function out of the solution to the discrete time problem, and then showed that this function converges to the solution of the continuous time problem, whereas we discretize the solution to the continuous time problem, and then show that the vector obtained solves the discrete time equations with a maximum error of $O(1/N)$. The result holds independently of the nature of the sampling functionals that yield $\bar{W}_i$ out of $W(\tau)$, that is, independently of how the underlying continuous process is sampled.

2. RESULTS

The symbol $\|f\|_\infty$ denotes the L_∞ norm of the function f, that is, the essential supremum of its absolute value,

$$\|f\|_\infty = \sup_x |f(x)|.$$

Let P be a partition of the interval $I = [0,1]$, that is, a finite set of points $\{x_i\}_{1\le i\le N}$ such that $0 \le x_1 < x_2 < x_3 \ldots < x_N \le 1$. Given a partition P and a function f one may consider the quantity

$$Q(P,f) = \sum_i |f(x_i) - f(x_{i-1})|,$$

which, for that given f, may or may not have an upper bound with respect to all finite partitions P. If it has such an upper bound we write $f \in BV$, and say that f is of bounded variation. The least upper bound of $Q(P,f)$ with respect to all finite P is the variation of f in $I = [0,1]$, denoted by $V(f)$.

Every continuous monotonic function defined on $I = [0,1]$ is of bounded variation, and its total variation is $|f(1) - f(0)|$. The continuity is not essential (any BV function has a finite derivative almost everywhere, that is, with the possible exception of a set of zero Lebesgue measure).

Consider the integral equation

$$W(\tau) + \Gamma A(\tau) + \int_0^T A(\sigma)W(\tau-\sigma)\, d\sigma = 0, \quad \tau \in [0,T],$$

where W (the covariance function) is known and A (the continuous time filter) is unknown (this is an example of a Fredholm integral equation of the second kind). The maximum value of the covariance function W is assumed at the origin, that is, $\|W\|_\infty = W(0)$. It is assumed that the parameter Γ satisfies $\Gamma > W(0)$, to ensure that the solution A to the integral equation exists. For an introduction to integral equations and functional analysis see [5, 7]. Two well-known classical references on the subject are [8, 9].

We recall the mean value theorem for integrals: given a continuous function f defined on $I = [a,b]$, there exists a point $\xi \in [a,b]$ such that

$$\int_a^b f(x)\, dx = f(\xi)(b-a).$$

Let us return to the prediction / smoothing problem and take, without loss of generality, $T = 1$ (recall that T is the duration of the observation interval). The relation $T = N\Delta$ becomes, of course, $\Delta = 1/N$.

We split the interval $I = [0,1]$ into N equal intervals I_k of length Δ. By the mean value theorem, there exists inside each interval I_k a point, denoted by ξ_k, such that, for any particular τ_i,

$$W(\tau) + \Gamma A(\tau) + \frac{1}{N}\sum_{k=0}^{N-1} A(\xi_k)W(\tau - \xi_k) = 0.$$

Note that the N points $\{\xi_i\}$ depend upon τ, and that the discretization of the integral is *exact*.

Assume now that we are given a set of N samples $A(\sigma_i)$ of $A(\tau)$ and that we wish to estimate the L_∞ norm of the difference between

$$\frac{1}{N}\sum_{k=0}^{N-1} A(\sigma_k)W(\tau - \sigma_k)$$

and

$$\int_0^T A(\sigma)W(\tau-\sigma)\, d\sigma.$$

Based on the previous discretization we reach the conclusion

$$\begin{aligned}
\left|\frac{1}{N}\sum_{k=0}^{N-1} A(\sigma_k)W(\tau-\sigma_k) - \int_0^T A(\sigma)W(\tau-\sigma)\,d\sigma\right| &= \\
\leq\ & \frac{1}{N}\sum_{k=0}^{N-1} |A(\sigma_k)W(\tau-\sigma_k) - A(\xi_k)W(\tau-\xi_k)| \\
=\ & \frac{V[A(\cdot)W(\tau-\cdot)]}{N} \\
=\ & \frac{V(A)W(0)+V(W)\|A\|_\infty}{N} \\
=\ & O(1/N),
\end{aligned}$$

assuming that A and W are of bounded variation. Note that although the points $\{\xi_k\}$ depended on τ, the final bound does not. Alternative versions of this result, in terms of the modulus of continuity of the integrand, could also be derived. Setting $\tau = i\Delta$ and $\sigma_k = k\Delta$ leads to

$$W(i\Delta)+\Gamma A(i\Delta)+\frac{1}{N}\sum_{k=0}^{N-1} A(k\Delta)W[(i-k)\Delta] = O(1/N).$$

However, this does not mean that the vector $A(k\Delta)$ is an approximate solution to the Levinson predictor equations

$$\bar{W}_i + \gamma\bar{a}_i^N + \sum_{j=0}^{N-1}\bar{W}_{i-j}\bar{a}_j^N = 0, \tag{3}$$

because sampling of the underlying continuous process must be performed using some kind of averaging pre-filter (directly sampling y would lead to a discrete-time process of infinite variance). To avoid this problem $W(i\Delta)$ should be replaced by some linear functional of W that preserves the uncorrelated nature of the signal and noise. Reference [10] uses

$$\bar{W}_{i-j} := \frac{1}{\Delta^2}\int_{j\Delta-\Delta}^{j\Delta}\int_{k\Delta-\Delta}^{k\Delta} W(t-s)\,dt\,ds,$$

but we will only assume that the functional leads to quantities $\tilde{W}$ satisfying

$$|\tilde{W}_{i-j} - W[(i-j)\Delta]| = O(1/N). \tag{4}$$

It follows from this relation that

$$\tilde{W}_i + \Gamma A(i\Delta) + \frac{1}{N}\sum_{k=0}^{N-1} A(k\Delta)\tilde{W}_{i-k} = O(1/N).$$

Comparing with (3) we see that the vector of samples $A(i\Delta)\Delta$ of the solution $A(\tau)$ to the continuous time problem is the solution $\bar{a}_i^N$ of the equations of the discrete time case, up to a residual of $O(1/N)$, provided that the sampling functional $\tilde{W}$, independently of its exact nature, satisfies (4).

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