JOINT BAYESIAN DETECTION AND ESTIMATION OF SINUSOIDS EMBEDDED IN NOISE

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ABSTRACT

In this paper we address the problem of the joint detection and estimation of sinusoids embedded in noise, from a Bayesian point of view. We first propose an original Bayesian model. A large number of parameters has to be estimated, including the number of sinusoids. No analytical developments can be performed. This lead us to design a new stochastic algorithm relying on reversible jump MCMC (Markov chain Monte Carlo). We obtain very satisfactory results.

1. INTRODUCTION

Spectral analysis of sinusoids embedded in noise is one of the most important problem in signal processing, due to its large number of applications. Bayesian statistical spectral analysis has been initiated by Jaynes and Bretthorst [4] and successfully applied. The crucial detection problem has been addressed in [4, 5]. However in those contributions, analytical approximations are performed. In this paper we propose an efficient stochastic algorithm which allows to obtain all the features of interest of the *a posteriori* probability distribution, which possesses all the information about the data including the dimension of the problem, without any analytical approximation.

2. BAYESIAN MODEL AND OBJECTIVES

This classical problem can be written in the following matrix form [5]:

$$\mathbf{y} = \mathbf{D}(\boldsymbol{\omega}_k)\,\mathbf{a}_k + \mathbf{n}_k \tag{1}$$

where $\mathbf{D}(\boldsymbol{\omega}_k)$ is the $N \times 2k$ matrix defined by $[\mathbf{D}(\boldsymbol{\omega}_k)]_{i,2j-1} = \cos[\omega_{j,k} i]$ and $[\mathbf{D}(\boldsymbol{\omega}_k)]_{i,2j} = \sin[\omega_{j,k} i]$ ($j = 1, \ldots, k$ and $i = 0, \ldots, N-1$), $\mathbf{a}_k \triangleq \begin{pmatrix} a^c_{1,k} & a^s_{1,k} & \ldots & a^c_{k,k} & a^s_{k,k} \end{pmatrix}^t$ are the amplitudes and $\boldsymbol{\omega}_k \triangleq \begin{pmatrix} \omega_{1,k} & \omega_{2,k} & \ldots & \omega_{k-1,k} & \omega_{k,k} \end{pmatrix}^t$ the radial frequencies of the k sinusoids. $\mathbf{y}$ is the observation vector containing N observations and $\mathbf{n}_k$ is an additive zero-mean white Gaussian noise with variance σ_k^2.These hypotheses on the noise can however be relaxed and will be presented in a future paper. We introduce notation $\boldsymbol{\theta}_k = \left(\mathbf{a}_k^t, \boldsymbol{\omega}_k^t, \sigma_k^2\right)^t$. Our objective is to estimate k and $\boldsymbol{\theta}_k$, given the observation $\mathbf{y}$.

In a Bayesian statistical framework one assumes that the parameters are distributed according to prior probabilities which express our degree of belief, or ignorance, for their different values. We first assume the following probabilistic structure, $p(\boldsymbol{\theta}_k) = p(\sigma^2)\,p(k, \mathbf{a}_k, \boldsymbol{\omega}_k)$, that is, the variance of the noise is independent of k and other parameters. Then, one can show that under the constraints $\Lambda \triangleq \mathbb{E}[k]$ and $\Xi \triangleq \frac{1}{N}\mathbb{E}\left[\mathbf{a}_k \mathbf{D}^t(\boldsymbol{\omega}_k)\,\mathbf{D}(\boldsymbol{\omega}_k)\,\mathbf{a}_k\right]$, the maximum entropy principle leads to the joint *a priori* probability density [1]:

$$\begin{aligned} p(k, \mathbf{a}_k, \boldsymbol{\omega}_k) \quad = \quad & \frac{\Lambda^k}{k!}\exp(-\Lambda)\,\frac{\left|\mathbf{D}^t(\boldsymbol{\omega}_k)\,\mathbf{D}(\boldsymbol{\omega}_k)\right|^{1/2}}{(2\pi\delta^2)^{k/2}} \times \\ & \times \exp\left[\frac{\mathbf{a}_k^t \mathbf{D}^t(\boldsymbol{\omega}_k)\,\mathbf{D}(\boldsymbol{\omega}_k)\,\mathbf{a}_k}{-2\delta^2}\right]\frac{\mathbb{I}_{\Omega_k}(\boldsymbol{\omega}_k)}{\pi^k} \end{aligned}$$

($\Omega_k \triangleq \left\{\boldsymbol{\omega}_k \in [0,\pi)^k ; 0 \leq \omega_{1,k} < \omega_{2,k} < \ldots < \omega_{k,k} < \pi\right\}$ is a set which removes identifiability problems and $\delta^2 \triangleq N\Xi/\Lambda$). We will see in Sect. 5 that these hyperparameters can be tuned in an objective way from the data. The parameter σ_k^2 is a scale parameter, and we thus assume that [3], $\sigma_k^2 \sim \mathcal{IG}\left(\frac{\nu_0}{2}, \frac{\gamma_0}{2}\right)$ which admits the Jeffreys' non-informative prior as a limiting case when $\nu_0, \gamma_0 \to 0$. ($x \sim \mathcal{D}(\cdot)$ means "x is distributed according to $\mathcal{D}(\cdot)$", $\mathcal{IG}(\cdot)$ is an inverted gamma, $\mathcal{N}(\cdot)$ a normal, $\mathcal{U}_A(\cdot)$ uniform distribution on A)

The posterior probability distribution is obtained from Bayes' theorem,

$$p(\boldsymbol{\theta}_k, k|\,\mathbf{y}) \propto p(\mathbf{y}|\,\boldsymbol{\theta}_k, k)\,p(\boldsymbol{\theta}_k, k) \tag{2}$$

from which it is not possible to obtain closed-form expression of: (**a**) the normalizing constant, (**b**) most of the marginal distributions, (**c**) the maximum *a posteriori* estimate. This stems from the fact that the parameter $\boldsymbol{\omega}_k$ introduces high non-linearity and that the dimension of the problem is not known. Indeed $p(\boldsymbol{\theta}_k, k|\,\mathbf{y})$ is defined on the union of subspaces, $\boldsymbol{\Theta} = \cup_{k=0}^{\lfloor N/2 \rfloor}\boldsymbol{\Theta}_k$ where $\boldsymbol{\Theta}_k \triangleq \mathbb{R}^+ \times \mathbb{R}^{2k} \times (0,\pi)^k$. This motivates the two following sections, where we provide a stochastic algorithm which allows to give a very satisfactory answer to those problems. In particular we address the problem of the estimation of k.

3. BASICS OF MCMC

MCMC are powerful stochastic algorithm which are briefly presented here. For details and many applications in statistical signal processing, see [2]. Roughly speaking, these methods consist in building an ergodic Markov chain $\left(\boldsymbol{\theta}^{(i)}_{k^{(i)}}\right)_{i\in\mathbb{N}}$, whose equilibrium probability distribution is the probability distribution of interest, say $\pi(\cdot)$. When this Markov chain has converged, under very mild sufficient conditions [7], ergodic theorems ensure that for any $\pi-$integrable function $f(\cdot)$, $\lim_{n\to+\infty}\frac{1}{n+1}\sum_{i=0}^{n} f\left(\boldsymbol{\theta}^{(i)}_{k^{(i)}}\right) = \mathbb{E}_\pi\left(f(\boldsymbol{\theta})\right)$. MAP value estimates can also be obtained. The most famous MCMC algorithms are the Metropolis-Hastings algorithm (MH) and the Gibbs sampler [7]. These algorithms, in their original form, do not allow to deal with dimension changes. Green [6] has provided a general framework to extend the MH algorithm to address the problem of dimension changes.

The principle of a MH step, is to update $\boldsymbol{\theta}_{k^{(i)}}^{(i)}$ at iteration i by making a proposition of move with a probability distribution $q\left(\boldsymbol{\theta}_{k^{(i)}}^{(i)}, d\boldsymbol{\theta}_{k^*}^*\right)$ and then accept this proposition $\boldsymbol{\theta}_{k^*}^*$ with an acceptance probability $\alpha = \min\left\{1, \frac{\pi\left(d\boldsymbol{\theta}_{k^*}^*\right) q\left(\boldsymbol{\theta}_{k^*}^*, d\boldsymbol{\theta}_{k^{(i)}}^{(i)}\right)}{\pi\left(d\boldsymbol{\theta}_{k^{(i)}}^{(i)}\right) q\left(\boldsymbol{\theta}_{k^{(i)}}^{(i)}, d\boldsymbol{\theta}_{k^*}^*\right)}\right\}$, or stay at $\boldsymbol{\theta}_{k^{(i)}}^{(i)}$. Reversibility is a key property of Markov chains, on which rely for example the classical MH algorithm, that easily ensures that a given probability distribution $\pi(d\cdot)$ is an invariant distribution of the Markov chain [7]. When considering a MH algorithm on a general state-space (including the union of subspaces, as in our case), with proposal distribution $q\left(\boldsymbol{\theta}_{k^{(i)}}^{(i)}, d\boldsymbol{\theta}_{k^*}^*\right)$ Green [6] has outlined that reversibility is automatically ensured when one is able to exhibit a symmetric dominating measure, for $\pi\left(d\boldsymbol{\theta}_{k^{(i)}}^{(i)}\right) q\left(\boldsymbol{\theta}_{k^{(i)}}^{(i)}, d\boldsymbol{\theta}_{k^*}^*\right)$ and $\pi\left(d\boldsymbol{\theta}_{k^*}^*\right) q\left(\boldsymbol{\theta}_{k^*}^*, \boldsymbol{\theta}_{k^{(i)}}^{(i)}\right)$ where $\boldsymbol{\theta}_{k^{(i)}}^{(i)} \in \boldsymbol{\Theta}_{k^{(i)}}$ and $\boldsymbol{\theta}_{k^*}^* \in \boldsymbol{\Theta}_{k^*}$. Whereas this measure is natural in fixed dimension problems, it must be built when model choice has to be performed. Green has provided a very general and flexible way for defining such a measure. First, complete the two subspaces with random variables $\mathbf{c}_1$,$\mathbf{c}_2$ such that $\dim(\mathbf{c}_1) + \dim\left(\boldsymbol{\theta}_{k^{(i)}}^{(i)}\right) = \dim\left(\boldsymbol{\theta}_{k^*}^*\right) + \dim(\mathbf{c}_2)$ and then define a reversible transformation $g_{k^{(i)},k^*}(\cdot)$ between the completed subspaces $\overline{\boldsymbol{\Theta}}_{k^{(i)}}$ and $\overline{\boldsymbol{\Theta}}_{k^*}$, such that $\left(\boldsymbol{\theta}_{k^*}^*, \mathbf{c}_2\right) = g_{k^{(i)},k^*}\left(\mathbf{c}_1, \boldsymbol{\theta}_{k^{(i)}}^{(i)}\right)$. The corresponding expression of the symmetric measure is given in [6], and allows to obtain densities of $\pi\left(d\boldsymbol{\theta}_{k^{(i)}}^{(i)}\right) q\left(\boldsymbol{\theta}_{k^{(i)}}^{(i)}, d\boldsymbol{\theta}_{k^*}^*\right)$ and $\pi\left(d\boldsymbol{\theta}_{k^*}^*\right) q\left(\boldsymbol{\theta}_{k^*}^*, \boldsymbol{\theta}_{k^{(i)}}^{(i)}\right)$ according to the same dominating measure [6]. Denoting $j_{a,b}$ the proposition of a move from $\boldsymbol{\Theta}_a$ to $\boldsymbol{\Theta}_b$, the acceptance ratio then writes $\alpha_{k^{(i)},k^*} = \min\left\{1, \frac{\pi\left(\boldsymbol{\theta}_{k^*}^*\right) q_2(\mathbf{c}_2) j_{,k^*,k^{(i)}}}{\pi\left(d\boldsymbol{\theta}_{k^{(i)}}^{(i)}\right) q_1(\mathbf{c}_1) j_{k^{(i)},k^*}} J_{g_{k^{(i)},k^*}}\right\}$ where $J_{g_{k^{(i)},k^*}}$ is the Jacobian of the transformation. Note that these reversible jumps are automatically defined by pair. A MH-Green update (and similarly its reverse movement) thus boils down to:

1. Choose movement from $\overline{\boldsymbol{\Theta}}_{k^{(i)}}$ to $\overline{\boldsymbol{\Theta}}_{k^*}$ with probability $j_{k^{(i)},k^*}$.
2. Draw $\mathbf{c}_1 \sim q_1(\cdot)$.
3. Evaluate $\left(\boldsymbol{\theta}_{k^*}^*, \mathbf{c}_2\right) = g_{k^{(i)},k^*}\left(\mathbf{c}_1, \boldsymbol{\theta}_{k^{(i)}}^{(i)}\right)$ and $\alpha_{k^{(i)},k^*}$.
4. Set $\boldsymbol{\theta}_{k^{(i+1)}}^{(i+1)} = \boldsymbol{\theta}_{k^*}^*$ with probability $\alpha_{k^{(i)},k^*}$ else $\boldsymbol{\theta}_{k^{(i+1)}}^{(i+1)} = \boldsymbol{\theta}_{k^{(i)}}^{(i)}$.

4. ALGORITHM

We first present the general scheme of the algorithm:

1. Initialization of $\boldsymbol{\theta}_{k^{(0)}}^{(0)}$ and $i = 0$.
2. Iteration i
3. Update, for $k^{(i)}$ given, the parameters $\mathbf{a}_k^{(i)}$, $\boldsymbol{\omega}_k^{(i)}$ and $\sigma^{2(i)}$.
4. Choose randomly between the following dimension changes:
   (a) "birth": add a sinusoid, at random.
   (b) "death": remove a sinusoid at random.
   (c) "split" randomly a sinusoid into two close sinusoids.
   (d) "merge" two close sinusoids.
5. $i \leftarrow i+1$ and Go to 2.

Step 3 is classical, and can be found in [1] for example. We now focus on the details of the birth-death and split-merge movements. Here the 4 movements are proposed with probabilities $b_k = s_k$, $d_k = m_k$ such as the definitions of b_k and d_k in [6].

The **birth-death** move is simple. The augmented states are **(a)** $\overline{\boldsymbol{\theta}}_{k+1} = \left(\mathbf{a}_{k+1}, \boldsymbol{\omega}_{k+1}, \sigma^2, \mathbf{c}_1 = (l^*, \mathbf{a}_k^*)\right) \in \overline{\boldsymbol{\Theta}}_{k+1}$ where l^* is the label of the victim and $l^* \sim \mathcal{U}_{1,\ldots,k+1}(\cdot)$, $\mathbf{a}_k^* \sim p\left(\mathbf{a}_k^* \middle| \boldsymbol{\omega}_{k+1\backslash l^*}, \sigma^2, \mathbf{y}\right)$ (it is a normal distribution, see [1] for its expression). **(b)** $\overline{\boldsymbol{\theta}}_k = \left(\mathbf{a}_k, \boldsymbol{\omega}_k, \sigma^2, \mathbf{c}_2 = (\omega^*, \mathbf{a}_{k+1}^*)\right) \in \overline{\boldsymbol{\Theta}}_k$ and $\omega^* \sim \mathcal{U}_{[0,\pi]}(\cdot)$, $\mathbf{a}_{k+1}^* \sim p\left(\mathbf{a}_{k+1}^* \middle| \boldsymbol{\omega}_k, \omega^*, \sigma^2, \mathbf{y}\right)$. The reversible transformation is here the identity.

The **split-merge** move is designed to solve overlapping problems. The augmented states are **(a)** $\overline{\boldsymbol{\theta}}_{k+1} = \left(\mathbf{a}_{k+1}, \boldsymbol{\omega}_{k+1}, \sigma^2, \mathbf{c}_1 = l_1^*\right) \in \overline{\boldsymbol{\Theta}}_{k+1}$ where $(l_1^*, l_2^* = l_1^* + 1)$ are the labels of the candidate sinusoids for the merge move, with $l_1^* \sim \mathcal{U}_{1,\ldots,k}(\cdot)$ **(b)** $\overline{\boldsymbol{\theta}}_k = \left(\mathbf{a}_k, \boldsymbol{\omega}_k, \sigma^2, \mathbf{c}_2 = (l'^*, u_0, u_1, u_2)\right) \in \overline{\boldsymbol{\Theta}}_k$ with $u_0 \sim \mathcal{U}_{(0,\pi)}(\cdot)$, $u_1 \sim \mathcal{N}\left(0, \frac{2\Lambda\delta^2}{N}\right)$, $u_2 \sim \mathcal{N}\left(0, \frac{2\Lambda\delta^2}{N}\right)$, $l'^* \sim \mathcal{U}_{1,\ldots,k}(\cdot)$. We note $E_i \triangleq a_{c_i}^2 + a_{s_i}^2$. The split-merge reversible function is defined here by (here for the split):

$$
\begin{aligned}
a_{c_{l_1^*}} &= u_1 \textit{ and } a_{c_{l_2^*}} = a_{c_{l'*}} - u_1 && (3)\\
a_{s_{l_1^*}} &= u_2 \textit{ and } a_{s_{l_2^*}} = a_{s_{l'*}} - u_2 && (4)\\
\omega_{l_1^*} &= \left[\omega_{l'^*} - u_0\sigma^*\sqrt{E_{l_2^*}/E_{l_1^*}}\right]\mathbb{I}_{(0,\pi)} && (5)\\
\omega_{l_2^*} &= \left[\omega_{l'^*} + u_0\sigma^*\sqrt{E_{l_1^*}/E_{l_2^*}}\right]\mathbb{I}_{(0,\pi)} && (6)
\end{aligned}
$$

where σ^* $(1/(10N)$ in our simulations) is a given parameter of the algorithm, which has no influence on the statistical model. The reverse move is automatically defined and one can easily check that the following equations are compatible with Eq. (3)-(6):

$$
\begin{aligned}
\omega_{l'^*} &= \left(E_{l_1^*}\omega_{l_1^*} + E_{l_2^*}\omega_{l_2^*}\right)/\left(E_{l_1^*} + E_{l_2^*}\right) && (7)\\
a_{c_{l'*}} &= a_{c_{l_1^*}} + c_{l_2^*} \textit{ and } a_{s_{l'^*}} = a_{s_{l_1^*}} + a_{s_{l_2^*}} && (8)
\end{aligned}
$$

We do not detail here the easy proof which ensures convergence of the algorithm and focus on simulation results.

5. SIMULATIONS

In practice, as outlined in Sect. 2, it is necessary to give Λ and Ξ values. In fact, only order of magnitudes for δ^2 and Λ are necessary. A solution to obtain those values is to consider δ^2 and Λ as parameters which are to be estimated. The Bayesian model is thus modified and one wants to estimate the following *a posteriori* probability distribution:

$$
\begin{aligned}
p\left(\delta^2, \Lambda, \boldsymbol{\theta}_k, k \middle| \mathbf{y}\right) \quad \propto \quad & p\left(\mathbf{y} \middle| \delta^2, \Lambda, \boldsymbol{\theta}_k, k\right) \times \\
& \times p\left(\boldsymbol{\theta}_k, k\right) p\left(\delta^2\right) p(\Lambda) \qquad (9)
\end{aligned}
$$

One gives δ^2 a diffuse *a priori* $\delta^2 \sim \mathcal{IG}(\alpha, \beta)$ $(\alpha = 2,\ \beta = \mathbf{y}^\dagger \mathbf{y})$. δ^2 can be easily updated as it is a conjugate prior [3]. For Λ the same method is applied and we assume a quasi non-informative *prior* [3] $\Lambda \sim \mathcal{IG}(1/2 + \varepsilon_1, \varepsilon_2)$ $(\varepsilon_i \ll 1\ i = 1, 2)$. In practice one easily obtains estimates of $p(\delta^2 | \mathbf{y})$ and $p(\Lambda | \mathbf{y})$, and thus approximate values for δ^2 and Λ. It is then sufficient to apply the algorithm presented in Sect. 4 with fixed values δ_0^2 and Λ_0 (*e.g.* the means), to estimate (2).

We present the two following experiments, similar to those found in [5]. The first one has the following parameters: $N = 64, k = 3$ and the sinusoids are described in Tab. 1

With $SNR = 10 \log_{10} E_1 / (2\sigma^2)$ as a definition, we have applied the algorithm to the three following situations $SNR_{1a} = 3db$, $SNR_{1b} = 0db$ and $SNR_{1c} = -3db$. We present the *a posteriori* probability distributions of the different hyperparameters in the different cases. We present estimations for $p(\delta^2 | \mathbf{y})$ and $p(\Lambda | \mathbf{y})$ only for SNR_{1b}. One notes that as the SNR gets weaker the energy is attributed to the wrong sinusoid. The second experiment has the following parameters: $N = 64, k = 2$ and the sinusoids are described in Tab. 2

With the definition given above, we performed the experiment for the following parameters, $SNR_2 = 3db$ is constant, and $l = 1, 4, 6, 8$. Due to lack of space, we do not present the Bayes factors [3], see [1].

Note the richness of the results, which could not be obtained from classical methods.

6. CONCLUSION

In this contribution, we have provided a Bayesian solution to the problem of joint detection and estimation of sinusoids embedded in noise. As closed-form expression for the posterior distribution is not possible, we propose a stochastic algorithm based on MCMC (Markov chain Monte Carlo) which allows to cleverly explore this posterior distribution. More precisely we use reversible jump MCMC introduced by [6] which allow spaces of different dimension to communicate and share redundant information. Results are very satisfactory, and the output of the algorithm provides a lot of information which can not be obtained from classical solutions. Furthermore, we show how it is possible to estimate satisfactory values for the hyperparameters, circumventing arbitrariness.

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i	E_i	$-\arctan(a_{s_i}/a_{c_i})$	$\omega_i/2\pi$
1	20	0	0.2
2	6.3246	$\pi/4$	$0.2 + 1/N$
3	20	$\pi/3$	$0.2 + 2/N$

Table 1. Parameters of the first experiment

i	E_i	$-\arctan(a_{s_i}/a_{c_i})$	$\omega_i/2\pi$
1	20	0	0.2
2	20	$\pi/4$	$0.2 + 1/(lN)$

Table 2. Parameters of the second experiment

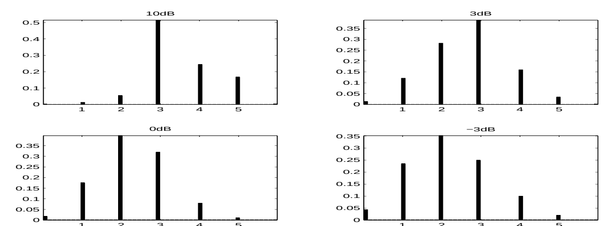

Figure 1. $\widehat{p}(k | \mathbf{y})$ **for** $4\ SNR$

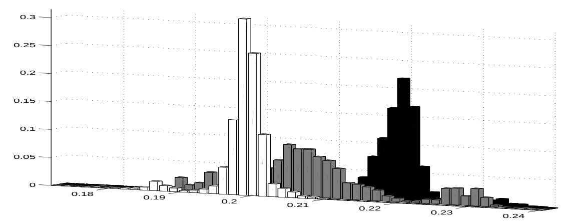

Figure 2. $\widehat{p}(\omega_i/2\pi | \mathbf{y}, k = 3)$ **for** $i = 1, 2, 3$ **and** SNR_{1a}

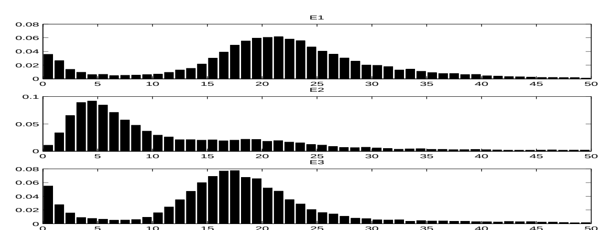

Figure 3. $\widehat{p}(E_i | \mathbf{y}, k = 3)$ **for** $i = 1, 2, 3$ **and** SNR_{1a}

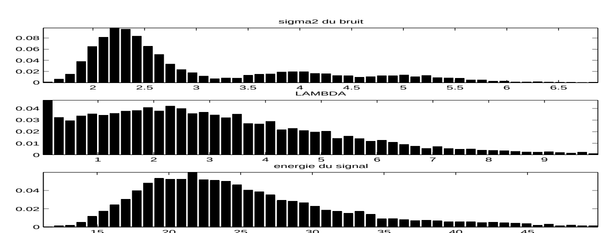

Figure 4. $\widehat{p}(\delta^2 | \mathbf{y})$, $\widehat{p}(\Lambda | \mathbf{y})$ **and** $\widehat{p}(\sigma^2 | \mathbf{y})$ **for** SNR_{1b}

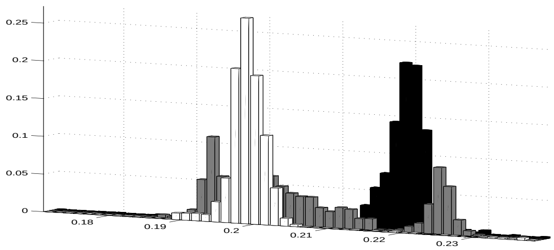

Figure 5. $\widehat{p}\,(\,\omega_i/\,2\pi|\,\mathbf{y},k=3)$ **for** $i=1,2,3$ **and** SNR_{1b}

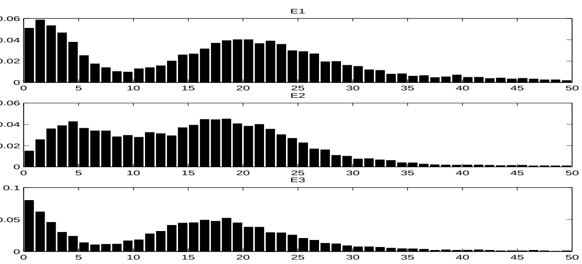

Figure 6. $\widehat{p}\,(\,E_i|\,\mathbf{y},k=3)$ **for** $i=1,2,3$ **and** SNR_{1b}

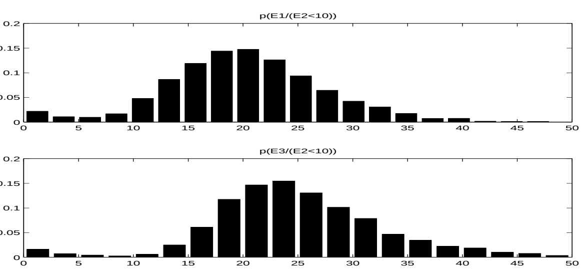

Figure 7. $\widehat{p}\,(\,E_i|\,E_2<10,k=3)$ **for** $i=1,3$ **and** SNR_{1b}

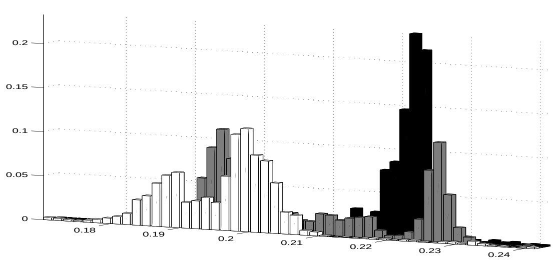

Figure 8. $\widehat{p}\,(\,\omega_i/\,2\pi|\,\mathbf{y},k=3)$ **for** $i=1,2,3$ **and** SNR_{1c}

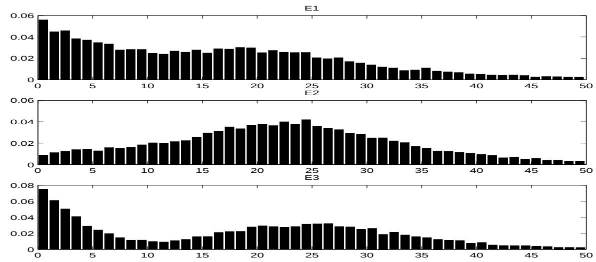

Figure 9. $\widehat{p}\,(\,E_i|\,\mathbf{y},k=3)$ **for** $i=1,2,3$ **and** SNR_{1c}

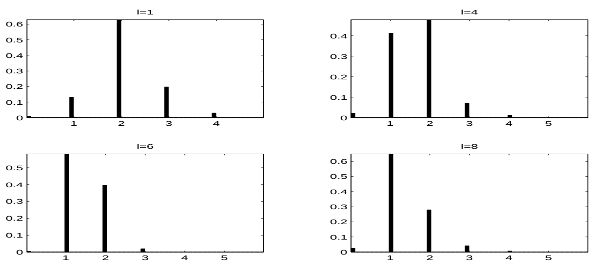

Figure 10. $\widehat{p}\,(\,k|\,\mathbf{y})$ **for** $l=1,4,6,8$ **and** SNR_2

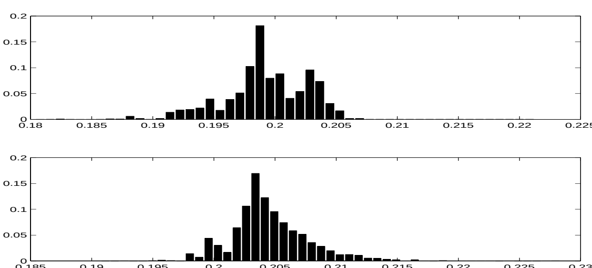

Figure 11. $\widehat{p}\,(\,\omega_i/\,2\pi|\,\mathbf{y},k=2)$ **for** $l=4$**,** $i=1,2$ **and** SNR_2

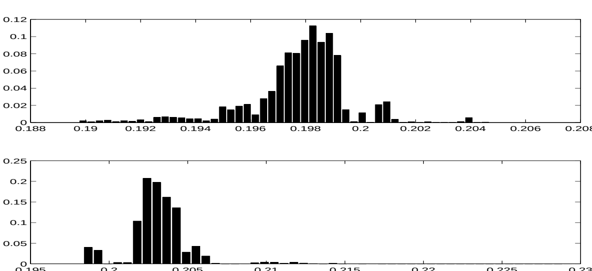

Figure 12. $\widehat{p}\,(\,\omega_i/\,2\pi|\,\mathbf{y},k=2)$ **for** $l=6$**,** $i=1,2$ **and** SNR_2