WREATH PRODUCTS FOR EDGE DETECTION

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ABSTRACT

Wreath product group based spectral analysis has led to the development of the wreath product transform, a new multiresolution transform closely related to the wavelet transform. In this work, we derive the filter bank implementation of a simple wreath product transform and show that it is in fact, a multiresolution Roberts Cross edge detector. We also derive the relationship between this transform and the two-dimensional Haar wavelet transform. We prove that, using a non-traditional metric for measuring edge amplitude with the wreath product transform, yields a rotation and translation invariant edge detector. We introduce a novel method for measuring the orientation of an edge and show that it is without error in the noise-free case. The wreath product transform edge detection performance is shown to be superior to many standard edge detectors.

1. INTRODUCTION

Edge detection is an important step in many image processing applications. The classic procedure for edge detection is to define an edge as an abrupt intensity change within an image. To detect the edge, a number of intensity gradients in different directions are computed, and from these the magnitude and orientation of the edge are extracted. Commonly, these gradients are computed using simple, well established operators such as the Sobel , Prewitt, and Roberts Cross [10] [2]. There are also many more sophisticated edge detectors available. The Canny edge detector [5] is proven to be optimal for many types of edges. Mallat et al. [6] have implemented a multiresolution Canny edge detector with a wavelet transform. Recently, other more complex wavelet-based multiresolution edge detectors have also been developed [4]. Multiresolution techniques are very well suited for edge detection, because as noted by Canny [5] and others, there is a tradeoff between noise removal and precise localization of an edge.

In this paper, we develop a multiresolution edge detector that uses the Roberts Cross edge detector as it's basis functions. This is done through the use of a wreath product transform (WPT).

This work follows recent work by Healy et al. [1] using finite group theoretic-based signal representation for image processing, aided further by the development of a fast FFT-based algorithm [3] for it's implementation. Using wreath product groups that arise as automorphism groups of spherically homogeneous trees, the spectral representation of these groups generates the wreath product transform, which is a multiresolution block transform that possesses the group-invariant property of the subspaces; that is, projections of functions onto subspaces remain within the subspaces, under group transformation on the underlying set on which the group acts.

After showing that the WPT is a multiresolution Roberts Cross edge detector, we show two ways to significantly improve the standard Roberts Cross performance. We will show that the magnitude measure given by the L_∞ norm is translation and rotation invariant. We also introduce a new metric for measuring edge orientation that is without error in the noise-free case. There have been many studies in the literature [10] [8] [7] that have evaluated the performance of various edge detectors. We will evaluate the WPT edge detector according to these criteria. Finally, we derive the relationship between the WPT and the two-dimensional Haar wavelet transform.

2. THE WREATH PRODUCT TRANSFORM

We select here the particular WPT originating from the homogeneous tree X_{4^9} [1]. Hence, given for example a 512×512 image, the image is divided into a 2×2 grid with the subgrids successively divided into 2×2 grids until blocks of size 2×2 are reached. We choose a quad tree indexing scheme where the pixels are scanned in a counterclockwise fashion, as shown in Figure 1, to generate a $512^2 \times 1$ vector. All groups are assumed to be the symmetric group S_4. For determining the first level of the WPT, a 4-point unitary one-dimensional discrete Fourier transform is applied to all successive 4-point sequences in the input vector.

$$\mathbf{V} = [\vec{V_0}, \vec{V_1}, \vec{V_2}, \vec{V_3}] = \mathbf{DFT}[X_0, X_1, X_2, X_3]. \qquad (1)$$

So by definition, our outputs are:

$$\begin{aligned} \vec{V_0} &= (X_0 + X_1 + X_2 + X_3)/2, & (2)\\ \vec{V_1} &= (X_0 - iX_1 - X_2 + iX_3)/2, & (3)\\ \vec{V_2} &= (X_0 - X_1 + X_2 - X_3)/2, & (4)\\ \vec{V_3} &= (X_0 + iX_1 - X_2 - iX_3)/2. & (5) \end{aligned}$$

The 4-point DFT is applied recursively to all lowpass outputs $\vec{V_0}$ at each stage, generating a pyramid type decomposition. We see that $\vec{V_0}$, $\vec{V_1}$, and $\vec{V_2}$ correspond to outputs

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$$\bullet X_0 \bullet X_3$$
$$\bullet X_1 \bullet X_2$$

Figure 1: Labeling convention for 2×2 pixel block

of lowpass, bandpass and highpass filters respectively, while $\vec{V_3}$ is just the complex conjugate of $\vec{V_1}$.

It is easily seen that the one-dimensional circular scanned DFT operation can also be implemented as a 4-band two-dimensional complex filter bank decomposition where the filters are as follows,

$$W_{LP} = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \tag{6}$$

$$W_{B1} = \frac{1}{2}\begin{bmatrix} -1 & -i \\ i & 1 \end{bmatrix}, \tag{7}$$

$$W_{HP} = \frac{1}{2}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}, \tag{8}$$

$$W_{B2} = \frac{1}{2}\begin{bmatrix} -1 & i \\ -i & 1 \end{bmatrix}. \tag{9}$$

The outputs are then downsampled, and the filter bank is recursively applied to the lowpass image. Looking at these filters we see that W_{B2} can be eliminated since it is the complex conjugate of W_{B1}. Assuming real data, we can separate filter W_{B1} into it's real and imaginary components, yielding the following four filters:

$$W_{LP} = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \tag{10}$$

$$W_{D1} = \frac{1}{2}\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}, \tag{11}$$

$$W_{D2} = \frac{1}{2}\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \tag{12}$$

$$W_{HP} = \frac{1}{2}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}. \tag{13}$$

Now we have a real filter bank implementation of a wreath product transform. We first note that $-W_{D1}$ and $-W_{D2}$ are the two gradient operators used by the Roberts Cross edge detector. Therefore, as these operators are recursively applied to the downsampled lowpass output, we see that the WPT is a multiresolution Roberts Cross edge detector. As an example of the wreath product spectrum, we see the magnitude after separation of subbands into real and imaginary components in Figure 2(a).

We note that the WPT operators are very similar to the basis function used by the two dimensional Haar wavelet transform. In Section 7, we will show that the two transforms are linearly related, and we derive the relationship.

3. EDGE DETECTION MODEL AND MEASURES

To evaluate the performance of the WPT edge detector, we will be using some standard models and measures that have become well established in the edge detection literature. Figure 2(b) shows a generalized model of the step edge that was presented by Abdou and Pratt [10]. Their model assumed that the edge was centered in a 2×2 block ($c = \frac{1}{2}$). We generalize this to allow the edge to lie anywhere within the block ($0 \le c \le 1$). As noted by Abdou et al. [10], because of the symmetry of both the edge model and the edge operators, tests need only be performed for edges that lie between $0°$ and $45°$.

The Roberts Cross edge detector uses a simple differential operator. The directional gradients ΔE_{D1} and ΔE_{D2} are computed in two orthogonal directions by convolving the image with the operators W_{D1} and W_{D2} as defined earlier. Therefore, for our sample pixel block in Figure 1,

$$\Delta E_{D1} = X_0 - X_2, \qquad \Delta E_{D2} = X_3 - X_1. \tag{14}$$

The orientation of the edge is then calculated as:

$$\phi = \tan^{-1}\frac{\Delta E_{D2}}{\Delta E_{D1}}. \tag{15}$$

We note ϕ is offset by $45°$ due to the gradient directions of the Roberts Cross operators. The magnitude of an edge is usually represented by the Euclidean or L_2 norm

$$M_2 = \sqrt{\Delta E_{D1}^2 + \Delta E_{D2}^2}. \tag{16}$$

Because of the complexity of computing the square root, L_1 and L_∞ norms are sometimes used instead,

$$M_1 = |\Delta E_{D1}| + |\Delta E_{D2}|, \tag{17}$$

$$M_\infty = max(|\Delta E_{D1}|, |\Delta E_{D2}|). \tag{18}$$

4. ROTATION INVARIANT EDGE DETECTION USING THE L_∞ NORM

The Sobel, Prewitt and Roberts Cross edge detectors using standard L_2 norm for magnitude measurement are not rotation invariant [10]. This has the effect that when thresholding the output to remove noise, any value chosen as a threshold will exclude edge points for some orientations while including non-edge points for other orientations.

Measures M_1 and M_∞ are typically employed only to reduce computation since for most edge operators, like the Sobel and Prewitt, they introduce significant errors [7]. Hence we observe a tradeoff between accuracy and complexity.

The interesting point is that this is not true for the Roberts Cross edge detector. Kitchen and Malin [8] in their comparative study of edge detectors, observed simulations which show that using the L_∞ norm to measure magnitude with the Roberts Cross operator produced orientation invariant measurements. Rosenfeld [9] has also shown that using the L_∞ norm measurement gives the magnitude of the "best-fitting" edge to a 2×2 pixel block.

We will show that using the L_∞ norm as our magnitude measure for the WPT, will yield an edge detector that does not have an orientation bias.

Assume we have an ideal edge of amplitude h as modeled in Figure 2(b) [10]. We see that the L_∞ measure gives the exact edge amplitude for any angle from $0°$ to $45°$. We apply the operators W_{D1} and W_{D2} and compute the L_∞ norm,

$$M_\infty = max(|X_0 - X_2|, |X_3 - X_1|), \tag{19}$$

where from the model, we see that

$$\begin{aligned} |X_3 - X_1| &= h, & (20)\\ |X_0 - X_2| &= |h(\tan\phi - 1)| \\ &\leq h & (21) \end{aligned}$$

which gives us,

$$M_\infty = h \tag{22}$$

Therefore we see that for an edge with a magnitude of h, the magnitude measured using the L_∞ norm will be h for any orientation ϕ and any location within the 2×2 pixel block as given by c. The L_∞ norm gives us a rotation and translation invariant edge detector.

5. A NEW METHOD TO OBTAIN AN UNBIASED ORIENTATION MEASUREMENT

The Sobel, Prewitt, and Roberts Cross edge detectors are not able to distinguish the orientation of an edge without error [10] [7]. Many researchers have tried various methods for correcting this bias, with varying degrees of success. Some ideas implemented are lookup-table correction factors, and an iterative procedure which modifies the magnitude of the operator masks depending on the value of the measured orientation [7]. We suggest a simple approach for the Roberts Cross edge detector. We introduce a new metric to determine the orientation. That is,

$$\phi_{WPT} = \tan^{-1}(1 - \frac{\Delta E_{D1}}{\Delta E_{D2}}), \tag{23}$$

where this measure matches the edge model. In comparing this to the traditional measure in equation (15), we see that the formulas are similar, wherein each takes a ratio of the two gradients.

Empirical tests show this new method gives the orientation of the model edge exactly, with no error.

6. PERFORMANCE OF THE WPT EDGE DETECTOR

Table 1 shows the comparative performance of some important edge detectors. We see that the WPT identifies the model edge without error. Also, we know the Roberts Cross is better able to localize an edge than the Sobel and Prewitt [10], given the smaller size of the operator. These are all positive features for the WPT.

The WPT is also a multiresolution transform and this is important since it can allow good control over the noise removal and localization tradeoff. We note that when performing edge detection at the lowest scale of the transform, edges are well localized, but are susceptible to noise. Any noise present in the image will show up in the L_∞ measurement, since there is no averaging at that scale. At the next higher scale, the noise will be reduced because of the averaging operation of the lowpass filter. The WPT is also easy to implement, given the small filter sizes.

EDGE DETECTOR	Angle Error (degrees)	Magnitude Error (percent)
Sobel	2.90	7.93
Prewitt	7.43	12.87
Roberts Cross	8.73	29.30
WPT	0	0

Table 1: Maximum Errors of Edge Detectors for $0° \leq \phi \leq 45°$. The Sobel, Prewitt, [7] and Roberts Cross magnitudes are with the standard L_2, ϕ metrics and the WPT magnitude is with the L_∞ and ϕ_{WPT} metrics.

7. THE RELATIONSHIP BETWEEN THE WPT AND THE HAAR WAVELET TRANSFORM

The WPT is closely related to the two dimensional Haar wavelet transform. The basis functions for the Haar are,

$$H_{LP} = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \tag{24}$$

$$H_{HE} = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}, \tag{25}$$

$$H_{VE} = \frac{1}{2}\begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}, \tag{26}$$

$$H_{HP} = \frac{1}{2}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}. \tag{27}$$

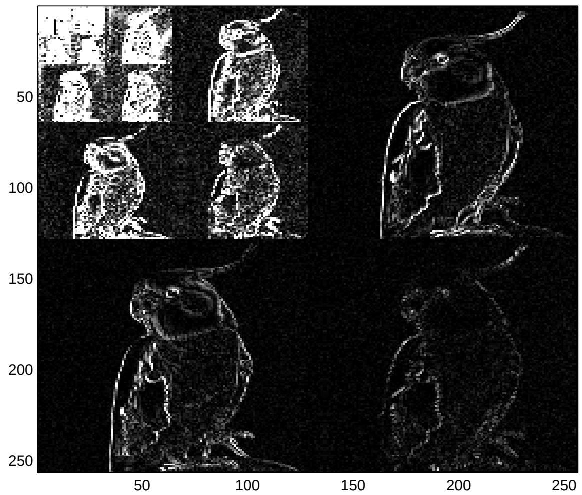
We can clearly see the linear relationship:

$$\begin{aligned} H_{LP} &= W_{LP}, & (28)\\ H_{HP} &= W_{HP}, & (29)\\ H_{HE} &= -(W_{D1} + W_{D2}), & (30)\\ H_{VE} &= W_{D2} - W_{D1}. & (31) \end{aligned}$$

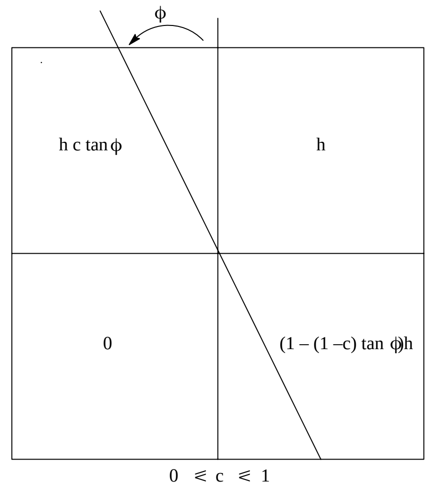
This leads to an interesting duality in their magnitude measures. Duality between the Roberts Cross edge detector and the Haar basis function was observed in simulations by Kitchen and Malin [8] and they concluded that the relationship was unexpected, and could not be trivially explained. But, using the relationships in equations (30) and (31), it becomes easy to extend the relationship to the norms. We apply the WPT operators W_{D1} and W_{D2} to the pixel block in Figure 1, and calculate the L_1, L_2, and L_∞ norms. We then apply the Haar operators H_{HE} and H_{VE} to the pixel block and compute their norms. Then after, some simplification, and applying the triangle inequality, we get

$$\begin{aligned} M_{1Haar} &= M_{\infty WPT} \times 2, & (32)\\ M_{\infty Haar} &= M_{1WPT}, & (33)\\ M_{2Haar} &= M_{2WPT} \times \sqrt{2}. & (34) \end{aligned}$$

Using these relationships, we see that we can obtain the same rotation invariant edge detection results with the Haar wavelet transform L_1 norm as with the WPT L_∞ norm. Singh et al. [11] and others have experimented with using Haar wavelets for edge detection, and our research serves to quantify their results and show methods to significantly improve the performance.



(a) Multiresolution Spectrum (real and imaginary)



(b) Edge Model

Figure 2: Multiresolution Spectrum and Edge Model

8. CONCLUSION

Using the WPT, we have developed a multiresolution Roberts Cross edge detector. We have shown how the use of the L_∞ norm yields a rotation and translation invariant edge measurement. We have also introduced a novel way to compute the orientation of an edge, that is error free for our edge model. We have shown how this yields an edge detector with significantly better performance than the Sobel, Prewitt, or conventional Roberts Cross edge detectors. Finally, by deriving the relationship between the Haar wavelet transform and the WPT, we have also provided additional insight into the properties of that classic transform.

9. REFERENCES

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