# **OPTIMAL SENSOR SCHEDULING FOR HIDDEN MARKOV MODELS**

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# ABSTRACT

Consider the Hidden Markov model where the realization of a single Markov chain is observed by a number of noisy sensors. The sensor scheduling problem for the resulting Hidden Markov model is as follows: Design an optimal algorithm for selecting at each time instant, one of the many sensors to provide the next measurement. Each measurement has an associated measurement cost. The problem is to select an optimal measurement scheduling policy, so as to minimize a cost function of estimation errors and measurement costs. The problem of determining the optimal measurement policy is solved via stochastic dynamic programming. Numerical results are presented.

# 1. INTRODUCTION

There are several signal processing applications where a variety of sensors are available for measuring a given process, however physical and computational constraints may impose the requirement that at each time instant, one is able to use only one out of a possible total of M sensors. There is also growing interest in flexible sensors such as multi-mode radar which can be configured to operate in one of many modes for each measurement. In such cases, one has to make the decision: Which sensor (or mode of operation) should be chosen at each time instant to provide the next measurement. It may also happen that one can associate with each type of measurement a per unit-of-time measurement cost, reflecting the fact that some measurements are more costly or difficult to make than others, although they may contain more useful or reliable information. The problem of optimally choosing which one of the M sensor observations to pick at each time instant is called the sensor scheduling problem. The resulting time sequence which at each instant specifies the best sensor to choose is termed the sensor schedule sequence.

Several papers have studied the sensor scheduling problem for systems with linear Gaussian dynamics where linear measurements in Gaussian noise are available at a number of sensors (see [1] for the continuous-time problem and [7] for the discrete-time problem). For such linear Gaussian systems, if the cost function to be minimized is the state error covariance (or some other quadratic function of the state), then the solution has a nice form: the optimal sensor schedule sequence can be determined a priori and is *independent* of the measurement data (see [1], [7] for details). This is not surprising; since the Kalman filter covariance is independent of the observation sequence.

In this paper we study the discrete-time sensor scheduling problem when the underlying process is a finite state Markov chain that is observed in white noise. The signal model is as follows: At each time instant, observations of a Markov chain in white noise are made at M different sensors. However, only one sensor observation can be chosen at each time instant. The aim is to devise an algorithm that optimally picks which single sensor to use at each time instant, in order to minimize a given cost function. We will show that *unlike the linear Gaussian case, the optimal sensor schedule in the HMM case is data dependent.* This means that past observations together with past choices of which observation to pick influence which observation to choose at present.

We have already mentioned that the papers [7] (discrete-time) and [1] (continuous-time) have considered the sensor scheduling problem for linear-Gaussian systems. We also draw the readers attention to [2] which treats the sensor scheduling problem for continuous time nonlinear systems. The general problem of stochastic control of partially observed finite-state Markov processes is treated in [10] (discrete-time) and [9] (continuous-time) as well as in the standard texts [6, 3]. We remark that while we use techniques from discrete-time stochastic control to solve the sensor scheduling problem, our problem is quite different in that the control (i.e. the sensor selection) does not enter into the dynamics of the Markov chain. In this sense the sensor scheduling problem is really an estimation problem and not a control problem.

The rest of the paper is organised as follows. In Section 2 we define the signal model followed in Section 3 with the formulation of the scheduling and estimation problem. In Section 4 we derive an equivalent fully observed stochastic control problem and give the relevant dynamic programming equations. In Section 5 we solve the dynamic programming equations numerically for some simple examples before concluding the paper and discussing some directions for further work.

Before we proceed we note that the purpose of this paper is to present the important problem of optimal sensor scheduling for hidden Markov models and to illustrate succinctly, the key ideas that lead to a solution of the problem via stochastic dynamic programming. This has necessitated a somewhat formal treatment of stochastic dynamic programming similar to the presentation in [3] which does not use the setting of measure theoretic probability. The main implication of this is that our probability measures and expectations are not mathematically well defined. We remark only that the ideas we present can be made mathematically rigorous without changing the main results. For a rigorous treatment of discrete-time stochastic dynamic programming, the reader is referred to [4] and [5, Chapter 10].

# 2. SIGNAL AND SENSOR MODELS

Let k = 0, 1, ... denote discrete time. Assume  $X_k$  is an S-state Markov chain with state space  $\{e_1, ..., e_S\}$ . Here  $e_i$  denotes the

S-dimensional unit vector with 1 in the *i*-th position and zeros elsewhere. This choice of using unit vectors to represent the state space considerably simplifies our subsequent notation. Let

$$a_{ji} = P(X_k = e_i | X_{k-1} = e_j), \quad i, j \in \{1, \dots, S\}$$

denote the transition probabilities of the Markov chain and let

$$\pi_0(i) = P(X_0 = i), \ i \in \{1, \dots, S\}$$

denote our a priori knowledge of the Markov chain. Write  $A = [a_{ji}]_{S \times S}$  and  $\pi_0 = [\pi_0(i)]_{S \times 1}$ .

Assume there are M noisy sensors available which can be used to give measurements of  $X_k$ . If the *i*-th sensor is chosen at time k, we make the scalar observation

$$y_k^{(i)} = H_k^{(i)} X_k + \Sigma_k^{(i)} X_k v_k^{(i)}, \quad i = 1, 2, \dots, M$$

Here  $H_k^{(i)}$  and  $\Sigma_k^{(i)}$  denote S element row vectors and  $v_k^{(i)}$  denotes white noise with known density  $f^{(i)}$ . We assume  $v_k^{(i)}$  is independent of  $v_k^{(j)}$ ,  $i \neq j$ .

At each time instant k, we are allowed to pick only one of the M possible sensor measurements  $y_k^{(i)}$ ,  $i \in \{1, \ldots, M\}$ . Also having picked this observation  $y_k^{(i)}$ , we are not allowed to look at any of the other M - 1 observations at time k.

A notationally convenient way of expressing this choice of picking one of M observations is again to use unit vectors. Let  $u_k$  denote a M state process; at each time it takes on one of M possible unit vectors  $e_1, \ldots, e_M$ . We will use this process  $u_k$  to denote which sensor to pick at time k. Then our choice of picking one sensor is equivalent to defining a new observation process  $y_k$ as follows

$$y_k(u_k) = \sum_{i=1}^M u'_k e_i y_k^{(i)}$$

Clearly if  $u_k = e_i$ , then  $y_k(u_k) = y_k^{(i)}$ .

We can now conveniently express the signal model as

$$y_{k}(u_{k}) = H_{k}(u_{k}) X_{k} + \Sigma_{k}(u_{k}) X_{k} v_{k}(u_{k})$$
(1)

where  $u_k \in \{e_1, \ldots, e_M\}$  and

$$H_{k}(u_{k}) = \sum_{i=1}^{M} u'_{k} e_{i} H_{k}^{(i)}$$
$$\Sigma_{k}(u_{k}) = \sum_{i=1}^{M} u'_{k} e_{i} \Sigma_{k}^{(i)}$$
$$v_{k}(u_{k}) = \sum_{i=1}^{M} u'_{k} e_{i} v_{k}^{(i)}$$

Remarks:

1. In the signal model (1) we have assumed that only one sensor is picked at each time. This is purely for convenience. It is straightforward to generalize the model to picking  $\overline{M}$  sensors (where  $\overline{M} < M$ ) at each time instant by merely increasing the dimension of  $u_k$  as follows: Define  $u_k = (u_k(1), \ldots, u_k(\overline{M})')$  where each  $u_k(i)$  is a unit vector. Then the signal model is identical to (1).

2. Note that  $v_k(u_k)$  is a white noise process. In particular, its sample path  $v_1(u_1), \ldots, v_k(u_k)$  is obtained by pasting together the appropriate sections of the sample paths of the white noise processes  $v_k^{(1)}, v_k^{(2)}, \ldots, v_k^{(M)}$ .

### 3. THE SCHEDULING/ESTIMATION PROBLEM

Let  $Y_k = \{u_1, u_2, \ldots, u_k, y_1(u_1), y_2(u_2), \ldots, y_k(u_k)\}$  so that  $Y_k$  represents the information available at time k upon which to base estimates and sensor scheduling decisions. The sensor scheduling and estimation problem proceeds in three stages for each  $k = 0, 1, \ldots, N - 1$ , where N is a fixed positive integer.

# 1) Scheduling:

Based on  $Y_k$  we generate  $u_{k+1} = \mu_{k+1}(Y_k)$  which determines which sensor is to be used at the next time step.

# 2) Observation:

We then observe  $y_{k+1}(u_{k+1})$  where  $u_{k+1}$  is the sensor selected in the previous stage.

### 3) Estimation:

After observing  $y_{k+1}(u_{k+1})$  we generate our next estimate of the state (or some function of the state) of the Markov chain:  $e_{k+1} = e_{k+1}(Y_{k+1})$ .

With these steps in mind, we define the sensor scheduling sequence

$$\mu = \{\mu_1, \mu_2, \ldots, \mu_N\}$$

and the estimator sequence

$$\epsilon = \{\epsilon_1, \epsilon_2, \dots, \epsilon_N\}$$

and say that the scheduler/estimator sequences are admissible if  $\epsilon_k$  is a function of  $Y_k$  and  $\mu_{k+1}$  maps  $Y_k$  to  $\{e_1, e_2, \ldots, e_M\}$ . Note that  $\mu$  and  $\epsilon$  are sequences of functions.

We assume that there is a cost associated with estimation errors and with the particular sensor schedule chosen. In particular, suppose that at each time, we wish to obtain estimates of some function of the Markov state  $s_k(X_k)$  where  $s_k : R^S \to R^{S'}$ . Let the estimation error be measured by the function  $g_k : R^{S'} \to R$ . Then we consider the cost function

$$\begin{aligned} J_{\mu,\epsilon} &= \mathbf{E} \{ \sum_{k=1}^{N} g_k(s_k(X_k) - \epsilon_k(Y_k)) \\ &+ \sum_{k=0}^{N-1} c_k(X_k, \mu_{k+1}(Y_k)) \} \end{aligned}$$

so that at each time the cost grows by an estimation error and per unit time sensor usage charge. Our aim is to minimise the total expected cost over all admissible scheduler/estimator laws.

The problem is greatly simplified by noting that the choice of estimator  $\epsilon_k$  does not impact on the future evolution of the system. This means the estimator optimization can be done independently at each time step. We now consider two examples:

#### 1) MMSE:

Let  $g_k(t) = t't$  so that the estimation criterion is the mean squared error. In this case it is well known that the conditional mean estimator is optimal so that we would choose  $\epsilon_k(Y_k) = \mathbf{E}\{s_k(X_k) \mid Y_k\}$ .

# 2) MAP:

Let  $g_k(t) = 1$  if  $t \neq 0$  and 0 otherwise. In this case the optimal estimator is the maximum a posteriori estimator  $\epsilon_k(Y_k) = s_k(e_{i^*})$  where  $i^* = \operatorname{argmax}_i P(X_k = e_i \mid Y_k)$ .

In the sequel we will concentrate on the MMSE case for which the cost functional becomes (note that it is no longer a function of  $\epsilon$  since the estimator law has been specified)

$$J_{\mu} = \mathbf{E} \{ \sum_{k=1}^{N} (s_k(X_k) - \hat{s}_k)' (s_k(X_k) - \hat{s}_k) + \sum_{k=0}^{N-1} c_k(X_k, \mu_{k+1}(Y_k)) \}$$
(2)

where

$$\hat{s}_k = \mathbf{E}\{s_k(X_k) \mid Y_k\} = \sum_{i=1}^{S} s_k(e_i) \ P(X_k = e_i \mid Y_k)$$

and our aim is to optimize  $J_{\mu}$  over the set of admissible control laws.

# 4. STOCHASTIC DYNAMIC PROGRAMMING FRAMEWORK

In this section we reformulate the optimal sensor scheduling problem as a fully observed stochastic control problem and give the dynamic programming equations that characterize the solution.

We begin by expressing the cost in terms of the conditional state probabilities (which constitute the *information state* for our problem),

$$\pi_k = [\pi_k(1), \pi_k(2), \dots, \pi_k(S)]'$$

where  $\pi_k(i) = P(X_k = e_i | Y_k)$ . Using the smoothing property of conditional expectation, the cost functional of (2) can be rewritten in the form

$$J_{\mu} = \mathbf{E}\{C_N(\pi_N) + \sum_{k=0}^{N-1} C_k(pi_k, \mu_{k+1}(\pi_k))\}$$
(3)

where

$$C_N(\pi_N) = \sum_{i=1}^{S} (s_N(e_i) - \hat{s}_N)'(s_N(e_i) - \hat{s}_N) \ \pi_N(i),$$

$$C_k(\pi, u) = \sum_{i=1}^{S} \left[ (s_k(e_i) - \hat{s}_k)'(s_k(e_i) - \hat{s}_k) + c_k(e_i, u) \right] \pi_k(i)$$

for  $k \in \{1, ..., N - 1\}$  and

$$C_0(\pi, u) = \sum_{i=1}^{S} c_0(e_i, u) \ \pi_0(i)$$

Note that since  $\pi_k$  is a sufficient statistic for  $Y_k$ , we have (without loss of generality) assumed that  $\mu_{k+1}$  is a function of  $\pi_k$  rather than  $Y_k$ .

Next we write down the recursive filter for the evolution of the state  $\pi_k$  (see [8]). First note that

$$\pi_{k+1}(i) = \frac{F(u_{k+1}, y_{k+1}(u_{k+1}), e_i) \sum_{j=1}^{S} a_{ji} \pi_k(j)}{\sum_{i=1}^{S} F(u_{k+1}, y_{k+1}(u_{k+1}), e_i) \sum_{j=1}^{S} a_{ji} \pi_k(j)}$$

where

$$F(u_k, y_k(u_k), e_i) = f^{(u_k)} \left( \frac{y_k(u_k) - H_k(u_k) e_i}{\Sigma_k(u_k) e_i} \right)$$

Writing

$$B(u_k, y_k(u_k)) = \operatorname{diag} \begin{pmatrix} F(u_k, y_k(u_k), e_1) \\ F(u_k, y_k(u_k), e_2) \\ \vdots \\ F(u_k, y_k(u_k), e_S) \end{pmatrix}$$

where  $\operatorname{diag}(t)$  is a diagonal matrix with t along the diagonal, we thus have

$$\pi_{k+1} = \frac{B(u_{k+1}, y_{k+1}(u_{k+1})) A' \pi_k}{\langle B(u_{k+1}, y_{k+1}(u_{k+1})) A' \pi_k, 1 \rangle}$$
(4)

where  $\langle \cdot, \cdot \rangle$  denotes the standard inner product in  $R^S$  and 1 represents an S-dimensional vector of ones. For ease of notation we express (4) as

$$\pi_{k+1} = T(\pi_k, u_{k+1}, y_{k+1}(u_{k+1})) \tag{5}$$

We now have a fully observed control problem: find an admissible control law,  $\mu$ , which minimizes the cost functional of (3), subject to the state evolution equation of (5).

## **Dynamic Programming Solution**

Based on the above formulation of the scheduling problem, the solution to the optimal sensor scheduling problem is obtained from the Dynamic Programming algorithm which proceeds backwards in time from k = N to k = 0:

$$J_N(\pi_N) = C_N(\pi_N)$$

and for 
$$k = N - 1, N - 2, ..., 0$$
  

$$J_{k}(\pi_{k}) = \min_{u_{k+1} \in \{1, ..., M\}} [C_{k}(\pi_{k}, u_{k+1}) + \int_{R} J_{k+1} (T(\pi_{k}, u_{k+1}, y_{k+1})) + (B(u_{k+1}, y_{k+1})A'\pi_{k}, 1) dy_{k+1}]$$
(6)

The optimal cost starting from the initial condition  $\pi_0$  is given by  $J_0(\pi_0)$  and if  $u_{k+1}^* = \mu_{k+1}^*(\pi_k)$  minimises the right hand side of (6) for each k and each  $\pi_k$ , the scheduling policy

$$\mu^* = \{\mu_1^*, \mu_2^*, \dots, \mu_N^*\}$$

is optimal.

We note that the state is now continuous valued and the dynamic programming equations involve an integral over the continuous valued observation space. A practical (suboptimal) algorithm can be obtained by discretising the state and observation spaces. This is the method we use to obtain the (approximate) optimal sensor scheduling policies for some simple examples in the following section.



Figure 1: Optimal Scheduling Policy for Scenario 1

### 5. NUMERICAL EXAMPLES

We consider a two state (S = 2) Markov chain with transition probabilities  $a_{11} = a_{22} = 0.8$  and  $a_{12} = a_{21} = 0.2$ . Observations are available from two sensors (M = 2) with  $H_k^{(1)} =$  $H_k^{(2)} = [0 \ 1]$ . The sensors differ in the noise scaling terms  $\Sigma_k^{(1)}$ and  $\Sigma_k^{(2)}$  and the costs of using the sensors per unit time of  $c_1 =$  $c_k(e_i, 1)$  and  $c_2 = c_k(e_i, 2)$ . We choose  $s_k(x) = x$  giving  $\hat{s}_k =$  $\pi_k$  and leading to an estimation error cost of  $1 - \pi'_k \pi_k$  at time k. Note that this cost is minimized when  $\pi_k = e_i$  for some i and maximise when all the conditional state probabilities are equal. The finite time horizon was set to N = 20. We consider two scenarios.

Scenario One: First, we model a situation when sensor one is significantly better than sensor two by choosing  $\Sigma_k^{(1)} = [0.1 \ 0.1]$ and  $\Sigma_k^{(2)} = \begin{bmatrix} 2.0 & 2.0 \end{bmatrix}$ . Not surprisingly when the costs of using each sensor are equal  $(c_1 = c_2)$  the optimal policy was to use sensor one all the time. As the cost of using sensor one is increased however, the optimal policy tends to select sensor one only when the state estimate is uncertain. An example is shown in Figure 1 for the case  $c_1 = 0.5$  and  $c_2 = 0.0$ . In this figure (and in Figure 2) the horizontal axis represents the time coordinate from k = 1 to k = N = 20. The horizontal axis is the conditional probability that the Markov chain is in state  $e_1$  at the time of interest. Note that since  $\pi_k(1) + \pi_k(2) = 1$  these figures fully specify the optimal scheduling policies. Note that within a few steps of the terminal time, the advantage of using sensor one in terms of improved state estimates, is not great enough to outweigh the cost of using the better sensor.

Scenario Two: In this scenario we have a situation where sensor one gives good measurements of the Markov chain when the chain is in one state and poor quality measurements when the chain is in the other state and vice versa for sensor two. We model this by setting  $\Sigma_k^{(1)} = [0.1 \ 2.0]$  and  $\Sigma_k^{(2)} = [2.0 \ 0.1]$  with  $c_1 = c_2$ . Given the symmetry of the model, it is not surprising that we obtain the optimal control policy shown in Figure 2 which says simply: use the sensor which gives good measurements of the state you are most likely to be in.

#### 6. CONCLUSIONS AND FUTURE WORK

In this paper we have tackled the optimal sensor scheduling problem for (finite-state) hidden Markov models using a stochastic dynamic programming framework. At this early stage, approximate



Figure 2: Optimal Scheduling Policy for Scenario 2

optimal scheduling policies can be derived by a brute force discretisation of the dynamic programming equations. Future work will look at efficient techniques for calculating optimal policies and detailed applications of these techniques to sensor management problems.

# 7. REFERENCES

- M. Athans. On the determination of optimal costly measurement strategies for linear stochastic systems. *Automatica*, 8:397–412, 1972.
- [2] J. S. Baras and A. Bensoussan. Optimal sensor scheduling in nonlinear filtering of diffusion processes. *SIAM J. Control* and Optimization, 27(4):786–813, July 1989.
- [3] Dimitri P. Bertsekas. Dynamic Programming and Optimal Control (Volume 1 and 2). Athena Scientific, Belmont, Massachusetts, 1995.
- [4] Dimitri P. Bertsekas and Steven E. Shreve. Stochastic Optimal Control: The Discrete-Time Case. Academic Press, New York, 1978.
- [5] Robert J. Elliott, L. Aggoun, and J. B. Moore. *Hidden Markov Models: Estimation and Control.* Springer-Verlag, New York, 1995.
- [6] P. R. Kumar and P. Varaiya. Stochastic Systems: Estimation, Identification and Adaptive Control. Prentice Hall, Englewood Cliffs, New Jersey, 1986.
- [7] Lewis Meier, John Peschon, and Robert M. Dressler. Optimal control of measurement subsystems. *IEEE Trans. Auto. Control*, 12(5):528–536, October 1967.
- [8] L. R. Rabiner. A tutorial on hidden Markov models and selected applications in speech recognition. *Proc. IEEE*, 77(2):257–285, February 1989.
- [9] Adrian Segall. Optimal control of noisy finite-state Markov processes. *IEEE Trans. Auto. Control*, 22(2):179–186, April 1977.
- [10] Richard D. Smallwood and Edward J. Sondik. The optimal control of partially observable Markov processes over a finite horizon. *Operations Research*, 21(5):1071–1088, 1973.