OPTIMUM OPEN EYE EQUALIZER DESIGN FOR NON-MINIMUM PHASE CHANNELS

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ABSTRACT

This paper contains results on the design of optimum equalizers to eliminate intersymbol interference in linear non-minimum phase channels conveying binary signals. The optimization is with respect to an open eye condition with a given delay. For causal stable channels with non-minimum phase zeros, we argue that this problem requires only the consideration of the FIR modified channel that has all the non-minimum phase zeros of the original channel. We show that if this modified channel can be equalized to yield an equalized system that is open eye with a specified delay, then the optimizing equalizer is, in fact FIR with all zeros outside the unit circle, and the impulse response of the equalised channel does not extend beyond the delay. We also give a simple necessary and sufficient condition to determine if for a particular delay, a given channel can be equalized to achieve an equalized response that is open eye.

1. INTRODUCTION

A classical objective in equalizer design for binary PAM is to achieve an open eye condition to alleviate Intersymbol Interference (ISI), [1]. The more open the eye pattern, the greater the margins against additive channel noise that cause errors. One way of viewing this condition is through figure 1, where $C(q^{-1})$ and $E(q^{-1})$ are the channel and the equalizer (both stable) respectively, q is the forward shift operator, and $u(k)$ is an input sequence that takes values from the set $\{-1, 1\}$. We say that the channel equalizer combination:

$$H(q^{-1}) = E(q^{-1})C(q^{-1}) = \sum_{i=0}^{\infty} h_i q^{-i} \qquad (1)$$

is *open eye with delay d* or *d-open eye* if for all binary $u(k)$, one has

$$Q[y(k)] = u(k-d),$$

where Q is the binary decision device obeying:

$$Q[a] = \begin{cases} 1 & if\ a > 0 \\ -1 & if\ a \leq 0. \end{cases}$$

This in turn is mathematically equivalent to the requirement that

$$\sum_{\substack{i=0 \\ i \neq d}}^{\infty} |h_i| < h_d, \qquad (2)$$

see any classical communication text book or [2]. Clearly, to reduce the possibility of errors, the mere satisfaction of (2) is not enough. Rather, it is desirable that

$$\frac{\sum_{\substack{i=0 \\ i \neq d}}^{\infty} |h_i|}{|h_d|} \qquad (3)$$

be minimized. In order to effect detection with small delay, it is also desirable to have as small a d as possible while keeping (3) small. Accordingly, this paper is concerned with minimization of (3) and the achievability of (2) for a given d.

To explain our results, we describe more precisely the setting of this paper. Should $C(q^{-1})$ be stable and minimum phase, then $E(q^{-1}) = C^{-1}(q^{-1})$ minimizes (3) with $d = 0$. If $C(q^{-1})$ has any zeros on the unit circle then (see Section 2) (2) is unachievable for any $d \geq 0$.

The remaining interesting case is when $C(q^{-1})$ has a mixture of zeros inside (stable zeros) and outside (unstable zeros) the unit circle. Since the stable minimum phase part of the channel can be cancelled by embedding its inverse in the equalizer, without loss of generality the minimization of (3) reduces to the following problem: *Design $E(q^{-1})$ to minimize (3) when $C(q^{-1})$ is a polynomial in q^{-1} of degree n_c with all its zeros (with respect to q) outside the unit circle.* It is this precise problem that we seek to address.

The main result of this paper provides the following surprising conclusion. Under these conditions, for a given d, should there exist an equalizer for which (2) holds, then the equalizer that minimizes (3) is FIR, in fact $d - n_c$ taps in length. Further, since a d-open eye system has precisely d zeros outside the unit circle, (see Section 2), this minimizing equalizer must have all its zeros outside the unit circle. Moreover, since a channel with n_c unstable zeros cannot satisfy (2) for $d < n_c$, this result also provides a simple check as to whether (2) can hold with $d = n_c$. Specifically, one must simply check if the nonminimum phase factor of the channel numerator is n_c-open eye. More generally, to check if an equalizer that achieves (2) exists, one must solve a single linear program.

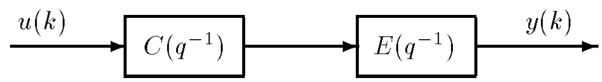

Figure 1: A channel equalizer combination

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Section 2 gives the preliminaries. Section 3 states the main results and explains its implications. Section 4 proves this result by using results from l_1 optimisation and duality theory. These have been used in a control system context in [4]. As an illustration of the main result Section 5 gives detailed formulas for channels with two real unstable zeros. Section 6 is the conclusion.

2. PROBLEM SETUP AND PRELIMINARIES

We first provide a result that relates the zero locations of a system to its ability to be open eye.

Theorem 2.1 *A causal stable system, $H(q^{-1}) = \sum_{i=0}^{\infty} h_i q^{-i}$, is d-open eye only if: (i) It has no zeros on the unit circle and (ii) precisely d zeros (including those at infinity) outside the unit circle.*

Proof: See for example [3]

∎

In view of the discussion in the introduction, we will work with the following standing assumption.

Assumption 2.1 *The channel*

$$C(q^{-1}) = \sum_{i=0}^{n_c} c_i q^{-i}$$

has all zeros finite and outside the unit circle. Further these zeros are distinct. Of these channel zeros, p are real and denoted $z_1, \ldots, z_p$. The channel also has $n-p$ conjugate pairs of zeros giving a total number of zeros $n_c = p + 2(n-p) = 2n - p$ denoted $z_1, \ldots, z_p, z_{p+1}, z_{p+1}^*, \ldots, z_n, z_n^*$ where superscript "" indicates complex conjugate.*

The assumption of distinct zeros could be dropped; the results given here will hold even without it, but with a more complicated proof. Further, it is easy to show that the assumption of finite valued zeros is without loss of generality.

To formulate the minimization of (3), take

$$E(q^{-1}) = \sum_{i=0}^{n_e} e_i q^{-i} \tag{4}$$

and set $N = n_c + n_e$, to obtain

$$H(q^{-1}) = C(q^{-1})E(q^{-1}) = \sum_{i=0}^{N} h_i q^{-i}. \tag{5}$$

Define,

$$\gamma(d, N) = \min_{E(q^{-1})} \frac{\sum_{\substack{i=0 \\ i \neq d}}^{N} |h_i|}{|h_d|}. \tag{6}$$

It is readily seen that for a given N, d, minimizing (6) is a linear program. In particular, by a simple scaling of $E(q^{-1})$ if need be, one can always choose $h_d = 1$. Further, the only constraint on the equalized system is that *its set of zeros must contain all the zeros of $C(q^{-1})$*. Then with $h = [h_0, \cdots, h_N]'$, $\gamma(d, N)$ is simply, the minimum of $\|h\|_1 - 1$, subject to the constraints that

$$\sum_{\substack{i=0 \\ i \neq d}}^{N} h_i z_l^{-i} = -z_l^{-d}, \quad \forall l \in \{1, \cdots, n\},$$

where the constraint for each complex zero also accounts for the constraint for its conjugate. Note: should $C(q^{-1})$ have multiple zeros, these constraints must be augmented by derivative relationships.

However, our goal is to optimize over *all possible N*. On the face of it, this would require solving an infinite number of linear programs of increasing dimensions. Our main result shows this to be unnecessary.

3. THE MAIN RESULT

Our main result is as follows.

Theorem 3.1 *Given a channel $C(q^{-1})$ satisfying assumption 2.1, with $\gamma(d, N)$ as in (6), there exists a unique positive integer $m \geq n_c - 1$ such that*

$$d > m \Rightarrow \gamma(d, N) = \gamma(d, d) < 1 \; \forall N \geq d; \tag{7}$$

$$d \leq m \Rightarrow \gamma(d, N) \geq 1 \; \forall N \geq d. \tag{8}$$

Furthermore, if $d > m$ then the optimizing equalized channel $H(q^{-1})$ has the form

$$H(q^{-1}) = \sum_{i=0}^{m} h_i q^{-i} + q^{-d} \tag{9}$$

where at most n_c of the $\{h_i\}_{i=0}^{m}$ are nonzero.

Before proving this result in Section 4 we discuss some of its implications. A channel can be equalized to give a d-open eye system if and only if $d > m$. A further consequence of (7) is that for given $d > m$ there is no reduction in (3) by using an equalizer of order greater than the minimal order necessary to make the system d-open-eye, namely $n_e = d - n_c$. In fact we have the following corollary:

Corollary 3.1 *Consider a channel $C(q^{-1})$ satisfying assumption 2.1. Suppose for a given d, a causal stable equalizer $E(q^{-1})$ that minimizes (3) subject to (1), achieves (2). Then this minimizing equalizer is FIR, of degree $n_e = d - n_c$. Further, all its zeros are unstable.*

Proof: Since the optimizing $H(q^{-1})$ of (9) is FIR of degree d and since any stable $H(q^{-1})$ inherits all n_c zeros of $C(q^{-1})$, the optimizing $H(q^{-1})$ is divisible by $C(q^{-1})$ to yield an equalizer $E(q^{-1})$ which is FIR of degree $n_e = d - n_c$. Because of Theorem 2.1 and assumption 2.1, all the zeros of this $d - n_c$-tap equalizer are unstable. ∎

The minimum allowable value of the delay d is $m + 1$. In general, computation of m is problematic. It can be arbitrarily large, even for a channel with a small number of nonminimum phase zeros. At the same time to preserve this as a finite problem, a bound given in [7] is useful. Explicit formulas for m are given in Section 5 for the case where the channel has only two, real, unstable zeros.

An important question to be addressed is given a delay d, can one find an equalizer that achieves (2)? The following corollary answers this question.

Corollary 3.2 *Under assumption 2.1, there exists $E(q^{-1})$ for which (2) holds, if and only if $\gamma(d, d) < 1$.*

Clearly checking whether $\gamma(d,d) < 1$, requires the solution of a finite order linear program. Further, should $\gamma(d,d) < 1$, then the equalizer achieving this $\gamma(d,d)$ also minimizes (3). Finally, since with $N = n_c$, $E(q^{-1})$ is a constant, n_c-open eyeness is achievable with a causal stable equalizer if and only if

$$|c_{n_c}| > \sum_{i=0}^{n_c-1} |c_i|.$$

4. PROOF OF THE MAIN RESULT

The proof of Theorem 3.1 is obtained by considering a problem which is the dual of the l_1 minimization problem of (6). The dual problem has a structure that is exploited to obtain the results. The connection between (6) and the dual version is via the following duality theorem, which is stated without proof. (See [5]).

Theorem 4.1 *If $A \in \mathcal{R}^{r\times s}$ and $b \in \mathcal{R}^r$, then:*

$$\min_{\substack{x\in l_1\\ Ax=b}} \|x\|_1 = \max_{\substack{\alpha\in R^r\\ \|A'\alpha\|_\infty \le 1}} \alpha' b$$

When applied to (6), the duality theorem yields the following.

Theorem 4.2 *Under assumption 2.1, for positive integers, N, d with $N \ge d \ge n_c$,*

$$\gamma(d,N) = \max_{\alpha\in\mathcal{R}^{n_c}} [-\alpha'\omega^d] \tag{10}$$

subject to

$$|\alpha'\omega^k| \le 1; \quad k = 0,1,2,\ldots,d-1,d+1,\ldots,N \tag{11}$$

where

$$\omega^k := (z_1^{-k},\ldots,z_p^{-k},\Re z_{p+1}^{-k},\Im z_{p+1}^{-k},\ldots,\Re z_n^{-k},\Im z_n^{-k}), \tag{12}$$

and $\Im$ denotes the "imaginary part".

Proof: In view of the discussion at the end of Section 2, one has that

$$\gamma(d,N) = \min_{g\in l_1} \|g\|_1 \tag{13}$$

subject to

$$\sum_{i=0}^{d-1} g_i z_j^{-i} + \sum_{i=d}^{N-1} g_i z_j^{-i-1} = -z_j^{-d}; \quad j = 1,2,\ldots,n, \tag{14}$$

where $g = [g_0,\ldots,g_{N-1}]'$, is related to h according to

$$\begin{aligned} g_i &= h_i; & i &= 0,1,\ldots,d-1,\\ g_i &= h_{i+1}; & i &= d,d+1,\ldots,N-1. \end{aligned}$$

The duality theorem can then be applied directly to give the desired result. ■

arise from the cost function
interpretation in the convex set hyperplane and a
We introduce some sets closely related to the feasible set (11):

$$S_l = \bigcap_{k=0}^{l} \{\alpha \in \mathcal{R}^{n_c} : |\alpha'\omega^k| \le 1\} \qquad \text{and} \qquad S = \lim_{l\to\infty} S_l.$$

Here ω^k is as in (12) so S_l is the set obtained from the first l constraints in (11) (including the "missing" d constraint if $l \ge d$) and S is the set obtained from infinitely many constraints (11). It is noted in [6] that S is compact. In addition we name the hyperplane pairs that form the boundaries of these sets:

$$H_k = \{\alpha \in \mathcal{R}^{n_c} : |\alpha'\omega^k| = 1\}.$$

The results of this paper depend on an understanding of which hyperplane pairs H_k intersect S. The following lemma shows that only a finite number of hyperplane pairs starting with H_0 and indexed by consecutive values of k intersect S.

Lemma 4.1 *There exists an integer m such that $H_k \cap S \ne \emptyset$ for $0 \le k \le m$ and $H_k \cap S = \emptyset$ for $k > m$.*

Proof:
The proof is in three parts:
1. $H_0 \cap S \ne \emptyset$.
2. There exists k such that $H_k \cap S = \emptyset$.
3. If $H_k \cap S = \emptyset$,then $H_{k+1} \cap S = \emptyset$.

Proof of 1. Consider the point $s_0 = [1,0,\ldots,0]' \in \mathcal{R}^{n_c}$. Note that $|s_0'\omega^0| = 1$, so $s_0 \in H_0$. Further, for all $k > 0$, since $|z_1^{-k}| < 1$,

$$|s_0'\omega^k| = |\Re z_1^{-k}| \le |z_1^{-k}| < 1.$$

This shows that $s_0 \in S$ also.

Proof of 2. Suppose that for all k, $H_k \cap S \ne \emptyset$. Let y_k denote an element in $H_k \cap S$. Since S is a compact set in $\mathcal{R}^{n_c}$, there is a number B such that $\|y_k\| \le B$. So, since $y_k \in H_k$,

$$1 = |y_k'\omega^k| \le \|y_k\|\|\omega^k\| \le B\|\omega^k\|.$$

Taking the limit $k \to \infty$ gives a contradiction since $\|\omega^k\| \to 0$.

Proof of 3. Suppose that

$$\eta = [\rho_1,\rho_2,\ldots,\rho_p,\mu_{p+1},\nu_{p+1},\ldots,\mu_n,\nu_n]'$$

is an element in H_{k+1}. Consider the vector $\sigma \in \mathcal{R}^{n_c}$ defined by

$$\sigma = [\rho_1 z_1^{-1},\rho_2 z_2^{-1},\ldots,\rho_p z_p^{-1},\tau_{p+1},\lambda_{p+1},\ldots,\tau_n,\lambda_n]'$$

where $\tau_i = \mu_i \Re z_i^{-1} + \nu_i \Im z_i^{-1}$, $\lambda_i = \nu_i \Re z_i^{-1} - \mu_i \Im z_i^{-1}$ and $i = p+1,\ldots,n$.

By writing each complex z_i in polar form and using the angle addition formulas for sine and cosine it is easy to check that for any non-negative integer r,

$$\sigma'\omega^r = \eta'\omega^{r+1}. \tag{15}$$

Since $\eta \in H_{k+1}$, $|\eta'\omega^{k+1}| = 1$. By (15), $|\sigma'\omega^k| = 1$, so $\sigma \in H_k$.

Since $H_k \cap S = \emptyset$, σ is not in S. This means that there exists an integer m such that $|\sigma'\omega^m| > 1$. By (15), $|\eta'\omega^{m+1}| > 1$, so η is not in S. Since η was an arbitrary element of H_{k+1}, $H_{k+1} \cap S = \emptyset$. ■

The result shown in Part 2 of the proof of Lemma 4.1 is known in the the l_1 optimal control context [4, 6]: after a certain terminal dual constraint, all subsequent constraints decay and have magnitude strictly less than one. Parts 1 and 3 taken together state that each constraint before this terminal constraint, contributes to the dual feasible set (i.e. none is redundant). Such a result appears to be new and is needed for the present problem. We are now ready to prove Theorem 3.1.

Proof of Theorem 3.1: Eqn (7): From Lemma 4.1, $S = S_m$, that is to say only hyperplane pairs $H_0, \ldots, H_m$ determine S. Denote by F the dual feasible set (11) for the dual problem (10,11). From (11) the boundaries of F are the N hyperplane pairs: $H_0, \ldots, H_{d-1}, H_{d+1}, \ldots, H_N$. Thus if $d > m$, then for any value of $N \geq d$, all of the hyperplane pairs $H_0, \ldots, H_m$ are boundaries of F so that $F = S_m = S$ and again by Lemma 4.1, $H_d \cap S = \emptyset$. Hence a cost of 1 is not attained at any point of F so that $\gamma(d, N) = \gamma(d, d) < 1 \forall N \geq d$.

Eqn (8): If $d \leq m$, then by Lemma 4.1, $H_d \cap S \neq \emptyset$. Hence there is a point on the feasible set F where the cost function takes on the value 1, so that $\gamma(d, N) \geq 1$ for any value of $N \geq d$.

Eqn (9): The fact that hyperplane pairs H_k for all $k > m$ do not contribute to the surface of F in the dual problem (10,11), corresponds with the solution to (13,14) having $g_k = 0$ for all $k > m$. Then if $d > m$, the minimization eqn (13,14) can be replaced by

$$\gamma(d, N) = \min_{g \in l_1} \|g\|_1 \tag{16}$$

where $g = [g_0, \ldots, g_m]'$, subject to

$$\sum_{i=0}^{m} g_i z_j^{-i} = -z_j^{-d}; \quad j = 1, 2, \ldots, n_c \tag{17}$$

where complex z_j are resolved into real and imaginary parts.

This is a standard scalar l_1 minimization problem and it is well known e.g. [7] that the optimal solution which we will call g_{opt} has at most n_c nonzero coefficients. The optimizing equalized response $H(q^{-1})$ is then

$$H(q^{-1}) = \sum_{i=0}^{m} (g_{opt})_i q^{-i} + q^{-d}, \tag{18}$$

which is the desired result.

5. RESULTS FOR CHANNEL WITH TWO REAL NONMINIMUM PHASE ZEROS

For the case where the channel has only two nonminimum phase zeros, the dual feasible set F is two dimensional and the dual maximisation for the calculation of $\gamma(d, N)$ can be carried out graphically to obtain closed form solutions. Here we present closed form solutions to the equalization problem for a channel with only two nonminimum phase zeros z_1, z_2, satisfying $1 < z_1 < z_2$. The following results were obtained using the approach in [8] and we note that it can be applied for any configuration of two real distinct zeros.

5.1. Closed Form Expressions

Firstly,

$$m = \underset{k \in \{1, 2, \ldots\}}{\text{argmin}} \frac{1 + z_2^{-k}}{z_1^{-k} - z_2^{-k}}. \tag{19}$$

For the case where $d > m$,

$$\gamma(d, N) = \frac{(1 + z_2^{-m}) z_1^{-d} - (1 + z_1^{-m}) z_2^{-d}}{z_1^{-m} - z_2^{-m}}. \tag{20}$$

The optimal equalised response is

$$H(q^{-1}) = \frac{z_1^{-d} z_2^{-m} - z_2^{-d} z_1^{-m}}{z_1^{-m} - z_2^{-m}} + \frac{z_1^{-d} - z_2^{-d}}{z_1^{-m} - z_2^{-m}} q^{-m} + q^{-d}. \tag{21}$$

q^{-1} of degree $(d-2)$.

5.2. Numerical Example

With $z_1 = 1/0.9$ and $z_2 = 1/0.3$, the channel is

$$C(q^{-1}) = (q^{-1} - 0.9)(q^{-1} - 0.3) = 0.27 - 1.2q^{-1} + q^{-2}.$$

Using eqn (19) gives $m = 3$. From Theorem 3.1, the channel can be made d-open-eye only for $d > 3$. With $d = 4$, eqn (21) gives the optimum equalized system:

$$H(q^{-1}) = 0.0618 - 0.9231q^{-3} + q^{-4}$$

and the corresponding equalizer is:

$$E(q^{-1}) = \frac{H(q^{-1})}{C(q^{-1})} = 0.0623 + 0.2769q^{-1} + q^{-2}.$$

6. CONCLUSION

In this paper we have applied results on l_1 optimisation and duality to the problem of designing channel equalizers which eliminate intersymbol interference in a linear discrete-time channel carrying binary signals. The main result is that for a given channel there is a lower bound $m + 1$ on the delay d which must be allowed before the signal can be resolved; that this delay of $m + 1$ is achievable; and that the optimum equalizer in this case yields an equalized response of length d.

We also show that the determination of whether (2) is achievable for a given delay d reduces to the solution of a simple linear program.

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