THE LOCAL MINIMA OF FRACTIONALLY-SPACED CMA BLIND EQUALIZER COST FUNCTION IN THE PRESENCE OF CHANNEL NOISE

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ABSTRACT

We study the local minima relocation of the fractionally spaced Constant Modulus Algorithm(FSE-CMA) cost function in the presence of noise. Local minima move in a particular direction as the noise power increases and their number may be eventually reduced. In such cases the performance of FSE-CMA may fail to adequately reduce inter symbol interference (ISI), but achieve an approximated MMSE by reducing its equalizer noise gain under certain constraints. We analyze the mechanism of relocation of FSE-CMA cost function local minima in terms of the auto-correlation matrix of sub-channel convolution matrix and its eigenvectors.

1. INTRODUCTION

This work concerning CMA [1] is based on the perfect equalization assumption of fractionally spacing with real Sub-Gaussian sources, i.e. sub-channel disparity and length condition [5], [4], [3], which allows identifying the combined channel-equalizer space with the equalizer space. We present a geometrical understanding of CMA cost function established on the combined space first, and utilize this framework for a noisy channel with real signals.

2. INTERPRETATION OF NOISE-FREE FSE-CMA COST FUNCTION

The CMA cost function can be regarded as a measure how much ISI a channel-equalizer combination can cause. We decompose CMA cost function by a radial and a spherical components from this view point.

Let $c := [c_0 \cdots c_k \cdots c_{N_c-1}]^T$ be a $T/2$-spaced channel and F be a real vector space consisting of equalizer taps,

$$F := \{f = [f_0 ... f_k ... f_{N_f-1}]^T \mid f_k \in R\} \cong R^{N_f}. \tag{1}$$

In a perfect FS-CMA case, it has been known that the combined channel-equalizer impulse response in the baud space h is given by a invertible matrix (sub-channel convolution matrix) transformation of the equalizer taps

$$h = Cf, \tag{2}$$

when c has no sub-channel common roots and $N_f = N_c - 2$ [3]. C is a $N_f \times N_f$ Sylvester matrix generated from the channel c. We identify the combined channel-equalizer space (or H-space) with the equalizer space F by the isomorphism given by C,

$$H \cong CF. \tag{3}$$

Now we introduce a measure of inherent ISI due to a tap vector on the channel-equalizer space H;

Definition 1 *For any vector $h = [h_0 \cdots h_{N-1}]^T \in H$,* the inherent ISI of h, *I_h is a real valued function on H defined by;*

$$I_h := \sum_{i \neq j}^{N-1} h_i^2 h_j^2 = \|h\|^4 - \sum_{i=0}^{N-1} h_i^4, \tag{4}$$

where $\|\cdot\|$ is the ℓ_2 norm of h.

Note that I_h has following properties;

- $0 \le I_h \le \|h\|^4 \frac{N-1}{N}$.
- For any $\|h\| > 0$, $I_h = 0$ if and only if h is a pure delay, i. e. . $h = [0 \cdots 0\ 1\ 0 \cdots 0]^T$.
- For any $\|h\| > 0$, $I_h = \|h\|^4 \frac{N-1}{N}$ when $|h_0| = |h_1| = , \cdots, = |h_{N-1}|$.

From these properties, we conclude that for the unit sphere $S^{N-1} := \{\|h\| = 1, h \in H\}$, I_h restricted to the sphere has $2N$ local minima (on each axis), $\{(0, .., 0, \pm 1, 0, .., 0)\}$, and 2^N local maxima on every $|h_0| = |h_1| =, ..., = |h_{N-1}|$ rays. The expansion of CMA cost function on H in the noise-free case will show how I_h is related to CMA cost function.

For a given identically independent, zero-mean, real source $s(n) = [s_n \cdots s_{n-N+1}]^T$ with the second central moment m_2, and the fourth central moment m_4, we define a statistical quantity, *the kurtosis deviation* of s as follows;

$$\eta_s := 3m_2^2 - m_4 = m_2^2(3 - \kappa_s), \tag{5}$$

where $\kappa_s = \frac{m_4}{m_2^2}$ is the kurtosis of s. Note that $\eta_s > 0$ for sub-Gaussian source (i.e. $\kappa_s < 3$) and $\eta_s = 0$ for a Gaussian source.

Since the equalizer output is given by $y_n = h^t s(n), \ \ h \in H$, we have

$$E(y^2) = m_2 \|h\|^2, \ \ E(y^4) = m_4 \|h\|^4 + \eta_s I_h. \tag{6}$$

The CMA cost function J_C can be seen as a real valued function on $H \cong R^N$ as;

$$\begin{aligned} 4J_C &= E\{(y^2 - \gamma)^2\} = E(y^4) - 2\gamma E(y^2) + \gamma^2 \\ &= m_4(\|h\|^2 - 1)^2 + \gamma^2 - m_4 + \eta_s I_h, \end{aligned} \tag{7}$$

where $\gamma = m_4/m_2$ is the CMA constant. For a sub-Gaussian source, from (7) we can interpret minimizing the noise-free FSE-CMA cost function as setting h at a certain "bumpy" sphere and minimizing I_h on that surface (See Figure 3). The spherical symmetry of $(\|h\|^2 - 1)^2$ term in (7) yields that J_C has local minima wherever I_h has, which results in making h a pure delay. We will generalize this approach for the noisy channel in following sections.

3. COST FUNCTION OF NOISY CHANNEL

Assume the real Gaussian white noise with variance σ^2 $w(k) = [w_k \cdots w_{k-N+1}]^T$ is uncorrelated with the source. Let $z(k) = [z_k \cdots z_{k-N+1}]^T$ be the colored noise of $w(k)$ through the equalizer f, $z_k = f^t w(k)$, and $z(n) = [z_n \cdots z_{n-N+1}]^T$ be the down sampled noise of $z(k)$, i. e. . $n = 2k + 1$ (Figure 1).

Then the output of the CMA equalizer is;

$$y_n = h^t s(n) + z_n. \tag{8}$$

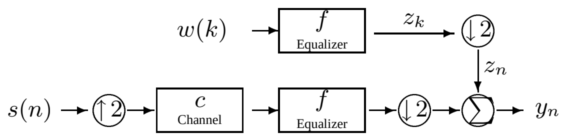


Figure 1. $T/2$**-spaced noisy channel model**

Simple calculations show

$$\begin{aligned} E(y^2) &= m_2\|h\|^2 + \sigma^2\|f\|^2 && (9) \\ E(y^4) &= E\{(h^t s)^4\} + 6E\{(h^t s)^2 z^2\} + E(z^4) \\ &= m_4\|h\|^4 + 6m_2\sigma^2\|h\|^2\|f\|^2 + 3\sigma^4\|f\|^4 && (10) \\ &\quad + \eta_s I_h + \eta_w I_f. \end{aligned}$$

Since w is Gaussian, $\eta_w = 0$. For the simplicity of calculation, we normalize the power of the source ($m_2 = 1$). Then $\gamma = m_4$ and using $f = C^{-1}h$, the CMA cost function becomes;

$$\begin{aligned} 4J_C &= \gamma\|h\|^4 + 6\sigma^2\|h\|^2\|C^{-1}h\|^2 + 3\sigma^4\|C^{-1}h\|^4 \\ &\quad -2\gamma(\|h\|^2 + \sigma^2\|C^{-1}h\|^2) + \gamma^2 && (11) \\ &\quad +\eta_s I_h. \end{aligned}$$

4. RADIAL MINIMA SURFACE

Due to the complexity of the cost function (11), we examine minima on every radial direction first and define a manifold Φ consisted of all such radial minima. On the surface we can determine where the CMA local minima locate. Unlike noise-free case [2], the manifold Φ is not spherically symmetric, which consequently causes the relocation of local minima in a certain direction.

Lemma 1 *For every ray on H-space, $R^+h := \{rh | r > 0\}$ (for a given $h \in S^{N-1}$), the CMA cost function restricted on the ray, $J_C|_{R^+h}$ has a unique minimum. Furthermore, these minima are located in the range of $(0, 1]$.*

Proof. The noisy FSE-CMA cost function (11) restricted on a ray R^+h' becomes;

$$\begin{aligned} 4J_C(rh) &= (3\sigma^4\|C^{-1}h\|^4 + 6\sigma^2\|C^{-1}h\|^2 + \gamma + \eta_s I_h)r^4 \\ &\quad -2\gamma(1+\sigma^2\|C^{-1}h\|^2)r^2 + \gamma^2. && (12) \end{aligned}$$

This is a quadratic function of r^2. The r satisfying $\partial J_C|_{R^+h}/\partial r = 0$ holds

$$r^2_{\min} = \frac{\gamma(1+\sigma^2\|C^{-1}h\|^2)}{3(1+\sigma^2\|C^{-1}h\|^2)^2 - \eta_s(1-I_h)} \qquad (13)$$

Notice that $0 < r_{\min} \leq 1$ as $\sigma^2\|C^{-1}h\|^2$ varies, because $0 < \eta_s(1-I_h) \leq \eta_s = 3-\gamma$. ■

Let φ denote the mapping from $h \in S^{N-1} \subset H$ to the scalar $r_{\min}$;

$$\begin{aligned} \varphi : S^{N-1} &\rightarrow \mathrm{R} && (14) \\ h &\mapsto r_{\min} \end{aligned}$$

Notice that φ is a positive differentiable function on S^{N-1}.

Let Φ be the union of all minima in the lemma 1,

$$\Phi := \{\varphi(h)h | h \in S^{N-1}\}. \qquad (15)$$

From differentiability of φ, Φ is a differentiable manifold homeomorphic to S^{N-1}. Every point on Φ is a candidate for a local minimum of cost function. We examine the possibility as follows:

Theorem 1 (The existence of local minima) *J_C has at least one pair of local minima at any noise power.*

Proof. Since Φ is a compact space (homeomorphic to a sphere), J_C has a minimum on Φ, which is also a local minimum of J_C on H. Since $J_C(h) = J_C(-h)$, we have a pair of local minima. ■

Lemma 2 *As noise power σ^2 increases, Φ-surface shrinks and deforms to a elliptic like shape (similar to $1/\|C^{-1}h\|^2_{S^{N-1}}$), if C is not a unitary matrix.*

Proof.

$$\varphi = \gamma\left(3(1+\sigma^2\|C^{-1}h\|^2) - \frac{\eta_s(1-I_h)}{(1+\sigma^2\|C^{-1}h\|^2)}\right)^{-1}$$

As $\sigma^2 \rightarrow \infty$, $\eta_s(1-I_h)/(1+\sigma^2\|C^{-1}h\|^2) \rightarrow 0$ and $3(1+\sigma^2\|C^{-1}h\|^2) \rightarrow \infty$. This results that $\varphi \rightarrow 0$ and Φ has shape like $1/\|C^{-1}h\|^2_{S^{N-1}}$. ■

Let $\rho(h)$ denote the output power of the CMA equalizer at h;

$$\rho(h) = E(y^2)|_h \qquad (16)$$

From (9), for $\forall \varphi(h)h \in \Phi$ ($h \in S^{N-1}$) we have

$$\begin{aligned} \rho(\varphi(h)h) &= (1+\sigma^2\|C^{-1}h\|^2)\varphi^2(h) \\ &= \frac{\gamma(1+\sigma^2\|C^{-1}h\|^2)^2}{3(1+\sigma^2\|C^{-1}h\|^2)^2 - \eta_s(1-I_h)}. && (17) \end{aligned}$$

Theorem 2 *On the Φ manifold, every local maximum with respect to the output power ρ is a local minimum of J_C and vice versa.*

Proof. From a direct calculation we have

$$\begin{aligned} 4J_C(\varphi(h)h) &= \gamma^2 - \frac{\gamma^2(1+\sigma^2\|C^{-1}h\|^2)^2}{3(1+\sigma^2\|C^{-1}h\|^2)^2 - \eta_s(1-I_h)} \\ &= \gamma^2 - \gamma\rho(\varphi(h)h). \end{aligned}$$

Since $\gamma > 0$, J_C is minimized when ρ is maximized. ■

From (17) recognize that for any CMA local minimum $h_{\min}$ it holds

$$\frac{\gamma}{3} < \rho(h|_{\min}) \leq 1\ . \qquad (18)$$

as shown in [6].

5. MINIMA RELOCATION IN A NOISY CHANNEL

Lemma 2 reduces the problem of finding minima of J_C on H-space to finding maxima of ρ on Φ. Intuitively, as noise power increases, if the fluctuation of the $1-I_h$ term in (18) can be ignored, the local maxima of ρ should appear where the $\|C^{-1}h\|$ is minimized. This results in the relocation of local minima of J_C from where the inherent ISI of the combined channel-equalizer (I_h) is minimized to where $\|C^{-1}h\|$ (or equivalently equalizer norm $\|f\|$) is minimized on the Φ (e. g. equalizer noise gain vs. ISI).

The function ρ restricted on Φ can be considered as a function S^{N-1} and rearranging (17) yields

$$\rho(h) = \gamma\left(3 + \eta_s\frac{I_h - 1}{(1+\sigma^2\|C^{-1}h\|^2)^2}\right)^{-1}, \quad h \in S^{N-1}. \qquad (19)$$

ρ can be seen as a function on $N-1$ dimensional disk by following identification as any function on S^{N-1};

$$h^0 := [h_1 \ldots h_{N-1}]^t \mapsto [\sqrt{1-\textstyle\sum_{i=1}^{i=N-1} h_i^2}\ h_1\ \cdots\ h_{N-1}]^T \quad (20)$$

for example,

$$(1-I_h)|_{S^{N-1}} = \textstyle\sum_{i=1}^{N-1} h_i^4 + (1 - \sum_{i=1}^{N-1} h_i^2.)^2 \qquad (21)$$

Canonically, $\frac{\partial}{\partial h^0}(\cdot)$ denotes the derivative of a function on S^{N-1} with respect to this identification, for example

$$\frac{\partial}{\partial h^0} I_h = \begin{bmatrix} h_1(h_0^2 - h_1^2) \\ \vdots \\ h_{N-1}(h_0^2 - h_{N-1}) \end{bmatrix} \quad (22)$$

Recognize that this identification depends on the choice of h_0 and can be considered on any arbitrary coordinate $g := Uh$, where U is a Unitary matrix, which will be denoted by g^0 and $\frac{\partial}{\partial g^0}(\cdot)$.

Let consider the derivative of ρ on S^{N-1} to find local minima location; Since the maxima location of ρ is the same as the minima location of $(I_h - 1)/(1+\sigma^2\|C^{-1}h\|^2)^2$, Setting $\frac{\partial}{\partial h^0}(I_h - 1)/(1+\sigma^2\|C^{-1}h\|^2)^2$ zero and multiplying by $(1+\sigma^2\|C^{-1}h\|^2)^3$ yield;

$$\frac{\partial}{\partial h^0} I_h + \sigma^2 \mathcal{N} = 0 \quad (23)$$

where we defined the noise term as $\mathcal{N}$;

$$\mathcal{N} := \|C^{-1}h\|^2 \frac{\partial}{\partial h^0} I_h - 2(I_h - 1)\frac{\partial}{\partial h^0}\|C^{-1}h\|^2. \quad (24)$$

Since we multiplied a positive quantity $(1 + \sigma^2\|C^{-1}h\|^2)^2$, any root of (23) with positive definite Hessian matrix is a local minimum of J_C. Recognize that (23) is a polynomial equation on a $N-1$ dimensional disk. We will exploit this property in following sections.

5.1. When there is no spherical relocation

Because $\|C^{-1}h\|^2 = h^t {C^{-1}}^t C^{-1} h$, we utilize the eigenvalue decomposition of the auto-correlation matrix of C^{-1}. Let $\lambda_0, ..., \lambda_{N-1}$ be the eigenvalues of ${C^{-1}}^t C^{-1}$ and $v_0, ..., v_{N-1}$ be the corresponding normalized eigenvectors such that

$${C^{-1}}^t C^{-1} = V \Lambda V^t,$$

where $\Lambda = \mathrm{diag}(\lambda_0, ..., \lambda_{N-1})$ and $V = [v_0 \cdots v_{N-1}]^T$.

If there is no eigenvalue disparity of ${C^{-1}}^t C^{-1}$, i.e. $\lambda_0 = \cdots = \lambda_{N-1}$, then $\frac{\partial}{\partial h^0}\|C^{-1}h\|^2 = 0$ and the equation (23) becomes

$$(1+\sigma^2)\frac{\partial}{\partial h^0} I_h = 0$$

This shows J_C has local minima wherever I_h has as the noise-free case, which implies noise causes no change of spherical local minima relocation except the radial shrinking due to noise in this case. Therefore, CMA works for reducing the inherent ISI of the combined channel-equalizer space in the absence of the eigenvalue disparity of the auto-correlation matrix of the sub-channel matrix. However, the physical meaning of these channels in frequency domain is not known yet as far as we know.

5.2. When Noise Changes the Number of Local Minima.

We now focus on the noise term, $\mathcal{N}$ of (23), since this term informs us of the asymptotic behavior of the equation as $\sigma^2 \to \infty$. Recognize that

$$\mathcal{N} = \|C^{-1}h\|^6 \frac{\partial}{\partial h^0} \frac{(I_h - 1)}{\|C^{-1}h\|^4}, \quad (25)$$

Thus the roots of $\mathcal{N} = 0$ with positive Hessian are the local minima of $\frac{(I_h-1)}{\|C^{-1}h\|^4}|_{S^{N-1}}$. To show that the eigenvalue disparity causes a reduction of the number of local minima of $\frac{(I_h-1)}{\|C^{-1}h\|^4}|_{S^{N-1}}$, we assume an extreme case that there is only one minimum eigenvalue which is significantly smaller than others.

Lemma 3 $\frac{(I_h-1)}{\|C^{-1}h\|^4}|_{S^{N-1}}$ **can** *have only two local minima on S^{N-1} near the minima location of $\|C^{-1}h\|_{S^{N-1}}$, when there is a unique minimum eigenvalue λ_m with large enough eigenvalue disparity ($\lambda_m \ll \lambda_i$).*

Proof. First, let assume C is a diagonal matrix. Rearranging $\mathcal{N}$ in componentwise yields

$$\frac{(\frac{\partial}{\partial h^0} I_h)_i}{4(I_h - 1)} - \frac{(\frac{\partial}{\partial h^0}\|C^{-1}h\|^2)_i}{2\|C^{-1}h\|^2} = 0, \;\; i = 1, ..., N-1. \quad (26)$$

Since we divided (26) by a negative quantity, $I_h - 1$, any root of (26) with negative Hessian will be a minimum. However, due to the complexity of computing Hessian we analyze the roots in a qualitative way by looking the shape of the two rational polynomials in (26). The first function is basically a scaled cubic polynomial. From a direct calculation we have $(\frac{\partial}{\partial h^0} I_h)_i / 4(I_h - 1) = h_i(c - 2h_i^2)/(I_h - 1) \le 1$ for all i, where c is a constant determined by $h_0, ..h_{i-1}, h_{i+1}, .., h_{N-1}$ (Figure 2). Meanwhile, because the second rational polynomial is related to a cross-section of a hyper ellipse in h_i direction, which is again a ellipse, it has a form of;

$$\frac{(\frac{\partial}{\partial h^0}\|C^{-1}h\|^2)_i}{2\|C^{-1}h\|^2} = \frac{(\lambda_i - \lambda_0)h_i}{\lambda_0 + (\lambda_i - \lambda_0)h_i^2 + a} \quad (27)$$

where a is a constant determined by $h_0, ..h_{i-1}, h_{i+1}, .., h_{N-1}$. This function has different shapes depending on the choice of λ_0 (Figure 2).

Type A If $\lambda_i \gg \lambda_0$, at $h_i = 0$ (minimum of the ellipse) it can be approximated as a line of which slope λ_i and near ± 1 at $h_i = \pm 1$.

Type B If $\lambda_i \ll \lambda_0$, at $h_i = 0$ (maximum of the ellipse) it can be approximated as a line of which slope -1 and monotonically decreasing nonlinear function otherwise.

Notice that type A determines a unique root of (26) with negative derivative, while type B can determine a unique root, of which derivative is positive, depending on an appropriate shift (Figure 2). Since the geometrical meaning of a root with negative Hessian is a roots with negative derivative in every direction, this implies we can have only one pair of minima candidate near minima of $\|C^{-1}h\|$ for large eigenvalue disparity of C.

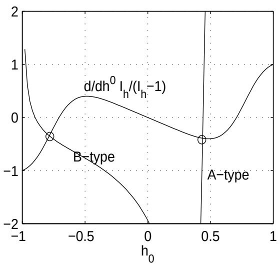
For a general C this can be also true in most case, although the cross-sected ellipse suffers from non-linear distortion (Figure 2). Since S^{N-1} is a compact space, we conclude this minima candidate should be the minima of $\frac{(I_h-1)}{\|C^{-1}h\|^4}|_{S^{N-1}}$. ■

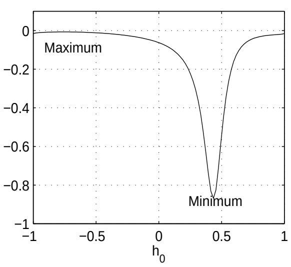
Theorem 3 *Finite noise* **can** *reduce the number of local minima of CMA when sufficiently large eigenvalue disparity of sub-channel convolution matrix exists.*

Proof. Rearrange (23) as follow;

$$\frac{1}{\sigma^2}\frac{\partial}{\partial h^0} I_h + \mathcal{N} = 0. \quad (28)$$

Suppose C holds the condition of Lemma 3. As $\sigma^2 \to \infty$ the term $\frac{1}{\sigma^2}\frac{\partial}{\partial h^0} I_h$ becomes flat and induces arbitrary small disturbance in $\mathcal{N} = 0$. Since every root x of $\mathcal{N} = 0$ can be considered as a intersection of two lines in a small neighborhood of x (Lemma 3), we can applies this disturbance to only a particular line. Then we can find a lower bound of $\hat{\sigma}^2$ such that $\frac{1}{\sigma^2}\frac{\partial}{\partial h^0} I_h$ term does not change any relative up-down position of the line, thus the sign of the Hessian of the roots, although it may shift the roots. Thus for any





a) Rational polynomials b) $\frac{(I_h-1)}{\|C^{-1}h\|^4}|_{S^{N-1}}$

Figure 2. Example of $\frac{(\frac{\partial}{\partial h^0} I_h)_i}{4(I_h-1)}$ and $\frac{(\frac{\partial}{\partial h^0}\|C^{-1}h\|^2)_i}{2\|C^{-1}h\|^2}$ for C in (29)

$\sigma^2 > \hat{\sigma}^2$, J_C has the same number of minima as $\frac{(I_h-1)}{\|C^{-1}h\|^4}|_{S^{N-1}}$, which is 2. Furthermore, the minima locate near the minima location of $\|C^{-1}h\|$. ∎

Intuitively, Theorem 3 says that because the CMA local minima tend to locate where minimizing the equalizer noise gain $\|f\| = \|C^{-1}h\|$ on Φ space than minimizing ISI for large noise power, the number of minima reduces when $\|C^{-1}h\|$ has only one pair of local minima. This result agrees with the result of [6] that "good" CMA local local minima are in the neighborhood of the Wiener solutions, and, at the same time, suggest that "bad" CMA local minima may disappear.

6. NUMERICAL EXAMPLES.

For an artificial $T/2$ channel $c = [-0.05,\ 1,\ 0.1,\ 0.5]^T$,

$$C = \begin{bmatrix} 1 & -0.05 \\ 0.5 & 0.1 \end{bmatrix}, {C^{-1}}^t C^{-1} = \begin{bmatrix} 16.6 & -31.7 \\ -31.7 & 64.2 \end{bmatrix}. \quad (29)$$

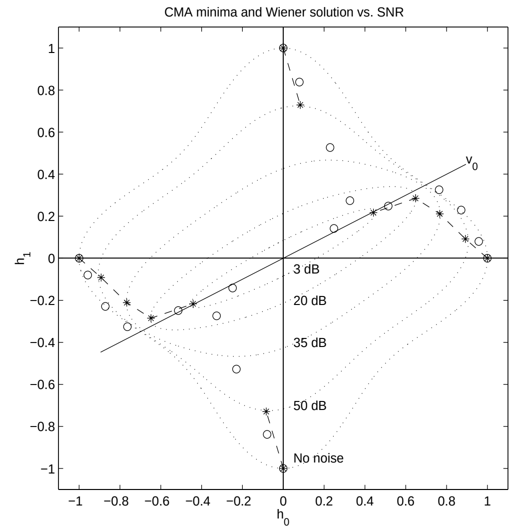
The eigen-decomposition of ${C^{-1}}^t C^{-1}$ is

$$\begin{array}{ll} \lambda_0 = 0.8, & v_0 = [0.8944\ \ 0.4472]^T \\ \lambda_1 = 80, & v_1 = [-0.4472\ \ 0.8944]^T \end{array}$$

Figure 3 shows the Φ surface (the dotted curves) deforming from a clover shape due to I_h to an elliptic shape due $\|C^{-1}h\|$ as noise power increases as well as its radial dilation. Corresponding to this change, the simulation result of the FSE-CMA (the '$*$' marks) shows the local minima move continuously toward v_0 direction and two of them disappear about 35 dB SNR. Thereafter with less than 35 dB SNR, we can have only 2 local minima in this channel. Meanwhile, Wiener solutions (the '$\circ$' marks) locate around CMA local minima.

7. CONCLUSION

We have presented a qualitative analysis of local minima relocation in a noisy channel. The noise power causes a radical shrink, which corresponds to reduction of equalizer noise gain, while the eigenvalue disparity of sub-channel convolution matrix is attributed to the spherical relocation. We determined the possible locations of local minima in the combined channel equalizer space in terms of Φ-surface at given noise power. It has been shown that in the presence of low SNR and large eigenvalue disparity the local minima tend to locate on Φ where minimizing the equalizer noise gain rather than minimizing ISI, which consequently results in the reduction of the number of CMA local minima.



Dotted curves	:	Φ surface
$*$	:	CMA local minima
$\circ$	:	Wiener Solution

Figure 3. A 2-dim. local minima relocation

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