SIGNAL RESTORATION WITH CONTROLLED PIECEWISE MONOTONICITY CONSTRAINT

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ABSTRACT

A signal restoration problem can be formulated as a least-squares inversion subject to a constraint that the signal has no more than k piecewise monotonic segments. We refer to the associated constraint as *controlled piecewise monotonicity* or CPM. We show that this constraint alone is powerful enough to stabilize an ill-posed inversion and enables us to incorporate the knowledge about the waveform geometry of the signal. This leads to a new algorithm for constrained signal restoration. We describe a highly efficient iterative scheme for computing the CPM constrained least-squares restoration. We also present experimental results and discuss issues related to the new algorithm.

1. INTRODUCTION

We consider a classic signal restoration problem:

$$\mathbf{y} = \mathbf{H}\mathbf{x} + \mathbf{e} \tag{1}$$

where $\mathbf{x}$ is the original signal which is distorted by a degrading operator $\mathbf{H}$ and additive noise $\mathbf{e}$, yielding an observed signal, $\mathbf{y}$. The task is to restore $\mathbf{x}$ from $\mathbf{y}$ with some knowledge about $\mathbf{H}$ and $\mathbf{e}$. In many cases, the degrading operator $\mathbf{H}$ is, or can be modeled as, a linear shift-invariant lowpass filter. Then the restoration of $\mathbf{x}$ based on the model of (1) is a "deblurring" type inverse problem which is notoriously ill-posed.

A popular approach to solving an ill-posed inverse problem is by regularization. Let us consider a regularized solution to (1) in discrete-time where $\mathbf{y}$ and $\mathbf{e}$ are $m \times 1$ vectors, $\mathbf{x}$ is an $n \times 1$ vector and $\mathbf{H}$ is an $m \times n$ Toeplitz matrix. Instead of solving (1) directly, the signal restoration problem can be recast as a functional minimization subject to some constraints

$$\mathbf{x}_\lambda : \quad \mathrm{Min}\left\{\Gamma_\lambda = ||\mathbf{y} - \mathbf{H}\mathbf{x}||^2 + \lambda ||\mathbf{C}\mathbf{x}||^2\right\} \tag{2}$$

where $||\cdot||$ denotes the ℓ_2 or least-squares norm, $\mathbf{C}$ is a regularizing operator, and $\lambda > 0$ is a parameter to be determined by matching specified constraints. (2) is known as the Tikhonov regularization[1] of the ill-posed inversion of (1). It is essentially a constrained least-squares (CLS) minimization that has a closed-form solution:

$$\mathbf{x}_\lambda = (\mathbf{H}^*\mathbf{H} + \lambda \mathbf{C}^*\mathbf{C})^{-1}\mathbf{H}^*\mathbf{y} \tag{3}$$

where $\mathbf{H}^*$ and $\mathbf{C}^*$ denote the Hermitian transpose of $\mathbf{H}$ and $\mathbf{C}$, respectively.

The Tikhonov regularization is achieved by smoothing. The regularizing operator $\mathbf{C}$ defines how a solution is smoothed, while the parameter λ controls the degree of smoothing. It is worth noting that when $\lambda \to 0$, (3) becomes the solution to an unconstrained least-squares (UCLS) minimization problem without any smoothing:

$$\mathrm{Min}\left\{\Gamma = ||\mathbf{y} - \mathbf{H}\mathbf{x}||^2\right\} \quad \text{with respect to } \mathbf{x}. \tag{4}$$

On the other hand, when $\lambda \to \infty$, smoothing is dominant. Smoothing is necessary for suppressing noise, but it also undermines one's capability of deblurring. In the Tikhonov regularization, the required amount of smoothing is proportional to the noise level. Consequently, one can achieve little of deblurring in a very noisy environment.

Another limitation of the Tikhonov regularization is associated with the regularizing operator $\mathbf{C}$. In general, $\mathbf{C}$ is chosen to be a highpass filter. Then the second term in the cost functional of (2) prescribes a penalty for excessive high frequency energy in the solution. This results in global smoothing which is undesirable in the regions where the signal has sharp transitions and/or high-frequency details.

Over the last two decades, there has been extensive research in iterative signal restoration. It was shown that many signal restoration techniques including the Tikhonov or the CLS method can be unified in a large family of iterative restoration algorithms[2, 3, 4]. More importantly, the iterative approach provides a flexible framework for one to incorporate *a priori* knowledge about the signal and noise in the form of multiple, linear or nonlinear constraints.

In this paper, we present a new algorithm for constrained iterative signal restoration. The innovation lies in a unique constraint that we call *controlled piecewise monotonicity*; we build a new form of CLS restoration based on this constraint. We show that this constraint alone is powerful enough to stabilize the ill-posed inversion and enables us to incorporate the knowledge about the waveform geometry of the signal. We also describe a highly efficient numerical algorithm for computing such constrained least-squares restoration and present experimental results.

2. LEAST-SQUARES RESTORATION SUBJECT TO CONTROLLED PIECEWISE MONOTONICITY

We repose the signal restoration problem as follows:

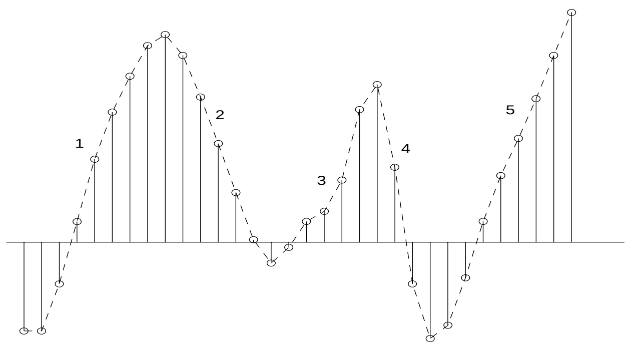

Figure 1: Piecewise monotonicity and local extrema. The signal shown here has 5 monotonic segments joined by 4 local extrema.

$$\mathbf{x}_k: \quad \text{Min}\left\{\Gamma_k = ||\mathbf{y} - \mathbf{H}\mathbf{x}||^2\right\} \text{ subject to the constraint that } \mathbf{x}_k \text{ has no more than } k-\text{piecewise monotonic segments.} \tag{5}$$

This is a nonlinear CLS problem. We refer to the associated constraint as *controlled piecewise monotonicity* (CPM). Before we attempt to solve this CLS problem, we examine what is meant by CPM and why it can be an appropriate constraint.

2.1. Piecewise Monotonicity as Model and Constraint

Most signals in the real world can be described as piecewise monotonic. Let us consider a discrete signal represented by a sequence $\{x[n]\}$ such as the one in Figure 1. This signal has 5 piecewise monotonic segments. Note that our definition of monotonicity is in non-strict sense. As such, our first monotonically increasing segment includes a constant part. In general, a signal may contain M piecewise monotonic segments, where M is a positive integer. We say a signal is $M-$*piecewise monotonic* if the signal has M piecewise monotonic segments. The points joining two consecutive monotonic segments are called *turning points*. It is easy to see that there are $M-1$ turning points in an $M-$piecewise monotonic signal, and they correspond to the local extrema of the signal.

Signals can be classified by the number of piecewise monotonic segments they contain. For example, all signals in $\mathbf{R}^n$ having no more than k piecewise monotonic segments can be grouped into a set that we denote by $\mathbf{Z}_k$. Clearly, $\mathbf{Z}_k$ is a collection of $M-$piecewise monotonic signals for $1 \leq M \leq k$. Note that the locations of the turning points are not specified, therefore, $\mathbf{Z}_k$ is a non-convex set.

The number of piecewise monotonic segments, M, provides a measure of fluctuation or oscillation in a signal. It is well-known that noise tends to be highly oscillatory. Indeed, a noisy signal often has a large M. This suggests that we may seek to limit the number of piecewise monotonic segments in a signal in order to control or reduce noise. More specifically, for a discrete signal $\mathbf{x} = \{x[n]\}$, we can constrain the signal to have no more than k piecewise monotonic segments. This is the CPM constraint in (5), which can now be expressed mathematically as requiring $\mathbf{x} \in \mathbf{Z}_k$. As we will show later, the CPM method proves to be very effective for denoising and stabilizing an ill-posed inversion. It is worth pointing out that denoising based on the CPM constraint is very different from a lowpass-filtering-type smoothing. For example, it is well-known that lowpass filtering results in ringing artifacts (Gibbs effect) near sharp transitions in a signal, such as a step edge. On the other hand, the CPM method prevents ringing from happening during denoising since ringing is as much oscillatory as (or more oscillatory than) noise.

The CPM method enables us to incorporate *a priori* knowledge about the signal. For example, we can prescribe k based on an estimated M of the signal from our knowledge about the waveform geometry of the signal. In some applications, such as MR spectroscopy, we often have a fairly good estimate of M. In more general cases, k may need to be computed separately or jointly in (5). In the following sections, we consider solving (5) with k given.

2.2. Least-Squares Piecewise Monotonic Approximation

As we have mentioned previously, the CPM constraint in (5) restricts the solution $\mathbf{x}_k \in \mathbf{Z}_k$. Since $\mathbf{Z}_k \subset \mathbf{R}^n$, for an arbitrary sequence $\mathbf{u} \in \mathbf{R}^n$, enforcing the CPM constraint can be accomplished by projecting $\mathbf{x}$ to $\mathbf{Z}_k$. Denote by $\mathcal{P}_{\mathbf{Z}_k}$ the projection operator, and $\mathbf{v}$ the orthogonal projection of $\mathbf{u}$ on $\mathbf{Z}_k$. $\mathcal{P}_{\mathbf{Z}_k}$ is defined under the ℓ_2 norm:

$$\mathcal{P}_{\mathbf{Z}_k}(\mathbf{u}): \quad \text{Min}\left\{||\mathbf{u} - \mathbf{v}||^2\right\} \text{ subject to } \mathbf{v} \in \mathbf{Z}_k. \tag{6}$$

It turns out that the key to implementing $\mathcal{P}_{\mathbf{Z}_k}$ is to compute a least-squares piecewise monotonic approximation to $\mathbf{u}$. This problem was studied in depth by Demetriou and Powell[5]. These authors developed a highly efficient algorithm for this purpose. Unfortunately, their work seems to be largely unknown to the signal processing community. Two recent applications of Demetriou and Powell's algorithm that we know of are due to the author of this paper and his associates[6, 7].

Assume the sequence $\mathbf{u}$ in (6) is M-piecewise monotonic. If $M \leq k$, $\mathbf{u} \in \mathbf{Z}_k$, therefore, $\mathbf{v} = \mathcal{P}_{\mathbf{Z}_k}(\mathbf{u}) = \mathbf{u}$. In general cases, $M > k$. The major task in computing $\mathbf{v}$ is to determine the optimal locations of a new set of turning points, $\{t_j : j = 1, ..., p\}$, $p \leq k$. Indeed, this appears to be a formidable combinatorial problem. Fortunately, much less work is actually needed because of a decomposition property of the optimal solution discovered by Demetriou and Powell[5]. This property allows the use of dynamic programming. Another important and interesting property of the optimal solution shows that the set of optimal turning points $\{t_j : j = 1, ..., p\}$ must be a subset of the local extrema of $\mathbf{u}$. This further reduces the amount of computation. Once the set of $\{t_j : j = 1, ..., p\}$ is determined, a least-squares monotonic approximation is computed independently for each segment. In the end, the use of Demetriou and Powell's algorithm for piecewise monotonic approximation has a complexity of $O(knM)$. We refer readers to the original work of Demetriou and Powell[5] for a detailed presentation of their algorithm. Figure 2 gives an example of denoising using Demetriou and Powell's algorithm.

2.3. Iterative Restoration with CPM Constraint

We compute a solution to the CLS restoration problem (5) using the following iterative algorithm:

$$\begin{aligned} \mathbf{x}_0 &= 0, \\ \mathbf{x}_j &= \mathcal{P}_{\mathbf{Z}_k}(\mathbf{x}_{j-1} + \alpha \mathbf{H}^*(\mathbf{y} - \mathbf{H}\mathbf{x}_{j-1})), \quad j \geq 1. \end{aligned} \tag{7}$$

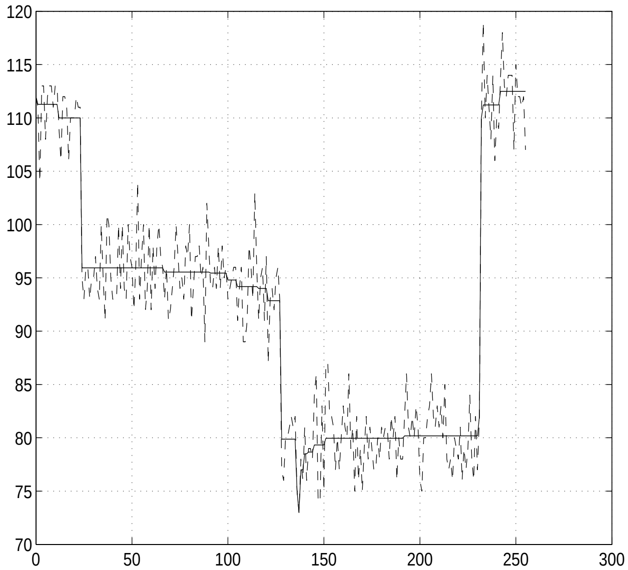

Figure 2: Denoising as piecewise monotonic approximation. A noisy signal (dashed line) is denoised (solid line) by using Demetriou and Powell's algorithm with $k = 2$.

where $0 < \alpha < 2/||\mathbf{H}^*\mathbf{H}||$. We give below some explanations and comments about the algorithm.

(7) shows that at each iteration the CPM constraint is enforced by $\mathcal{P}_{\mathbf{Z}_k}(*)$ which projects the sequence in $(*)$ to $\mathbf{Z}_k$. Some readers may have already recognized that if we drop $\mathcal{P}_{\mathbf{Z}_k}$, (7) will become the Van Cittert iteration that converges to the minimum-norm solution to the unconstrained least-squares inversion (4). The Van Cittert-type algorithm, also known to some researchers as Bialy or Landweber iteration, has been studied extensively[2, 3, 4]. Our algorithm combines the CPM constraint with the Van Cittert iteration to solve the CLS problem (5). Since the Van Citter iteration is convergent with an appropriate choice of α, and the projection operation $\mathcal{P}_{\mathbf{Z}_k}(*)$ is non-expansive, the convergence of algorithm (7) is assured.

3. EXPERIMENTS

We tested our algorithm with a large number of simulated signals. In the experiments, we blurred the signal with some commonly encountered degrading operators such as the Gaussian. We also added noise to the blurred signal. Then, we used algorithm (7) with a prescribed k to restore the signal. To assess the quality of a restoration, we define the signal-to-noise ratio (SNR):

$$\mathrm{SNR} = 20 \log_{10} \left(\frac{||\mathbf{u}||}{||\mathbf{u} - \mathbf{v}||} \right) \quad \mathrm{dB} \tag{8}$$

where $\mathbf{u}$ is the original signal and $\mathbf{v}$ is a distorted or restored signal.

Figure 3 shows the results from one of the experiments. The original signal in Figure 3(a) is composed of segments of linear and exponential functions. It has 8 piecewise monotonic segments. The signal was blurred by a Gaussian with a standard deviation $\sigma = 4$ and added with noise. The distorted signal is shown in Figure 3(b); it has a SNR of 9.96 dB. Figure 3(c) shows a signal restored by algorithm (7) with $k = 10$ for 100 iterations; the improved SNR is 20.78 dB. The restored signal in Figure 3(d) was obtained in the same way except that $k = 8$ was used; the improved SNR is 21.17 dB.

4. DISCUSSION

It appears to us that the CPM constraint alone is sufficient to stabilize the deblurring-type inversion in a very noisy environment. The number k plays a critical role in the restoration. Excellent results are obtained when the prescribed k is close to the actual number of piecewise monotonic segments that the original signal has. When k is larger than necessary, we observe some spurious artifacts (see Figure 3(c)). This is because by prescribing a larger k we accept a higher degree of fluctuation in the signal. Consider an unconstrained inverse filtering. It can be completely dominated by noise. When we use the CPM constraint with a larger k, we essentially allow certain part of noise to grow. Nonetheless, the noise is prevented from exploding. It is conceivable that if k is too small, some features (i.e., peaks) in the signal may be wiped out. In practice, it may be hard to get a perfect k, but one can often get into a right range by trials. It is also possible to include k as a variable to be optimized in (5).

The CPM constraint can be used in other types of signal restoration and reconstruction problems. For example, we have successfully used it to stabilize bandlimited extrapolation as well as reconstruction of a signal from its wavelet extrema representation[7]. We are very interested in generalizating this paradigm to 2-D for solving image restoration problems. Intuitively, the CPM constraint in 2-D may be quite appropriate because an image is naturally segmented into piecewise smooth regions by edges. It follows that a CPM constraint may be used to limit the number of regions or the number of edges in an image. Unfortunately, a straightforward generalization from 1-D is not available because monotonicity in 2-D needs to be associated with an orientation on a surface. It is even more ambiguous to speak of piecewise monotonicity for a surface.

5. REFERENCES

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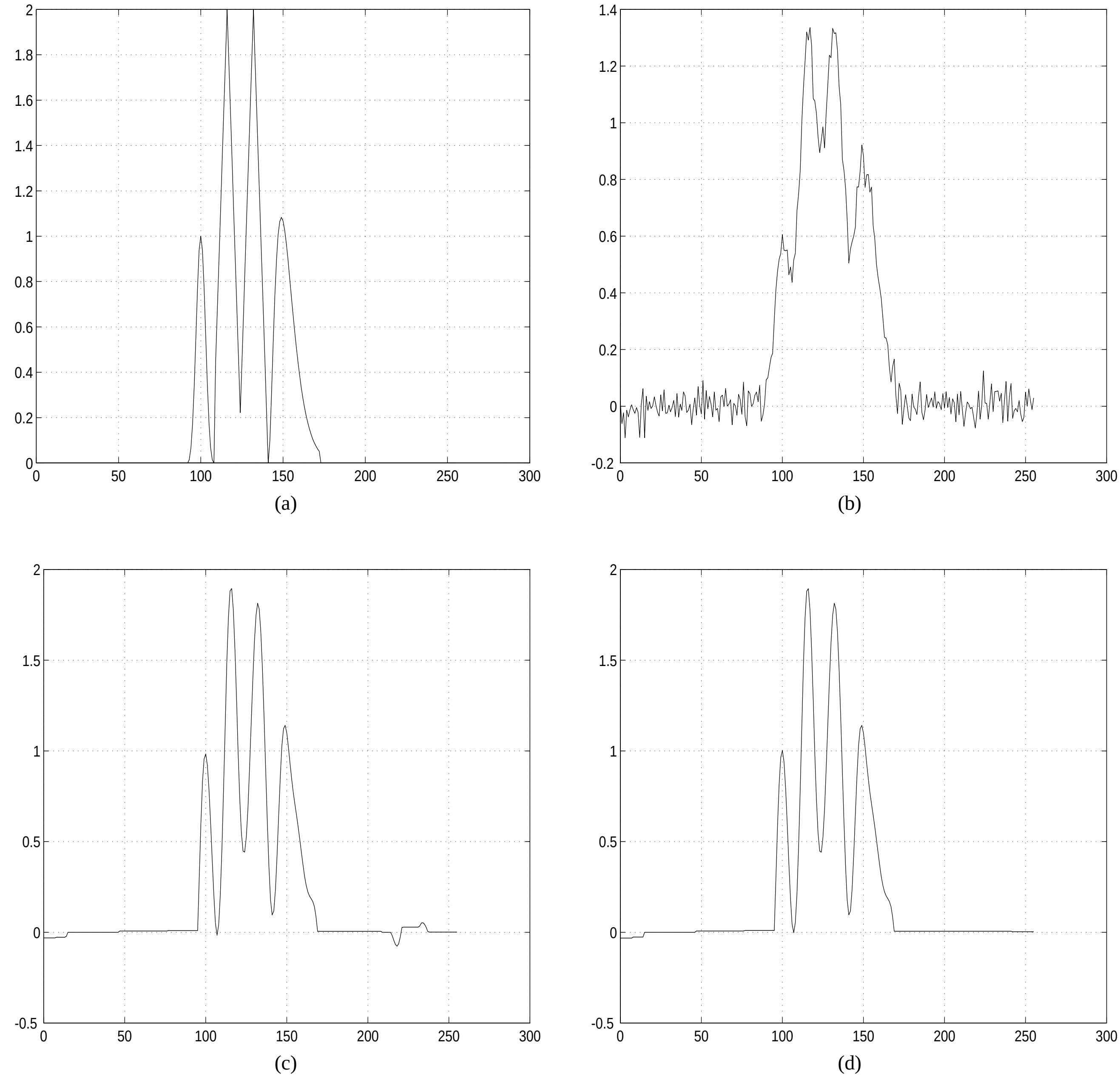

Figure 3: Signal restoration with the CPM constraint. (a) A simulated signal having 8 piecewise monotonic segments; (b) blurred by a Gaussian ($\sigma = 4$) and added with noise, SNR=9.96 dB; (c) restored using algorithm (7) with $k = 10$ for 100 iterations, SNR=20.78 dB; (d) same as in (c) but with $k = 8$, SNR=21.17 dB.