SMOOTH ORTHONORMAL WAVELET LIBRARIES: DESIGN AND APPLICATION

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ABSTRACT

For signal-based design of orthonormal (ON) wavelets, an optimization of a cost function over an N-dimensional angle space is required. However: (1) the N-dim space includes both smooth and non-smooth wavelets; (2) many of the smooth wavelets are similar in shape. A more practical approach for some applications may be to construct a library of smooth ON wavelets in advance—a library that consists of representative wavelet shapes for a given filter length. Existing ON wavelet libraries (Daubechies, nearly-symmetric, Coiflets) provide only one wavelet for each filter length. We construct ON wavelet libraries using local variation to determine wavelet smoothness and the discrete inner product to discriminate between wavelet shapes. The relationship between library size and the similarity threshold is investigated for various filter lengths. We apply an entropy-based wavelet selection algorithm to an example signal set, and investigate compactness in the wavelet domain as a function of library size.

1. INTRODUCTION

Recently, compactly supported orthonormal (ON) wavelets have been parametrized [10], allowing signal-based wavelet design to be performed by optimization of a cost function in the parameter space. The parametrization of [10] links a wavelet of support $[-N, N+1]$ to a set of N angles; *any* choice of the N angles leads to a valid orthonormal wavelet basis. Several different algorithms for ON wavelet design based on optimization of a cost function have been proposed [2, 8]. Any optimization is complicated by the fact that the angle space includes both smooth and non-smooth wavelets since the parametrization guarantees only a vanishing zeroth moment. In most applications, however, at least some degree of smoothness is required [3, 6, 7].

Furthermore, in many matching procedures, especially those performed on a window by window basis [5], it may be desirable to sacrifice small improvements in compactness of representation to an improvement in efficiency (most optimization routines usually cannot guarantee the global extremum anyway). Small changes in the angle values result in small changes in wavelet shape (for smooth wavelets): two wavelets whose parametrization angles are close together will result in similar decomposition coefficients for a given signal. Also, as we will show, the nature of the angle parametrization is such that wavelets of similar shapes can be found at widely differing angles.

Rather than performing an optimization for each input signal, then, a more practical approach may be to construct in advance a library of smooth ON wavelets. Matching to signal(s) can then be done by selecting the wavelet from the library that optimizes some measure of interest, such as entropy of the wavelet coefficients [1]. Existing ON wavelet libraries (Daubechies, nearly-symmetric, Coiflets) provide only one wavelet for each filter length. It is of interest, then, to determine what kinds and how many smooth wavelets of distinct shapes can be found in the infinitely large collection of ON wavelets.

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2. THE DISCRETE WAVELET TRANSFORM

The Discrete Wavelet Transform (DWT) can be computed with the filterbank in Fig. 1 in which a sequence of interest is decomposed into detail sequences at different resolutions (scales), $d_1, d_2, \ldots, d_J$ and a coarse approximation sequence c_J.

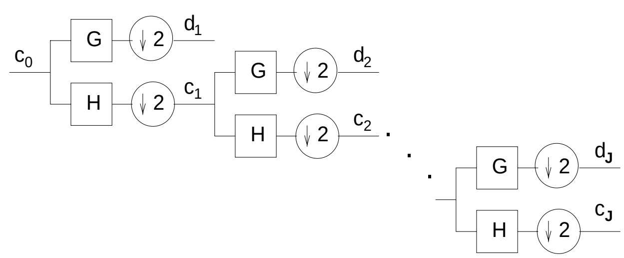

Figure 1: A J-level analysis filterbank.

The basis functions of a practical DWT are not shifted and scaled versions of the wavelet and the scaling function. Rather, they are shifted versions of the discrete-time wavelet sequences $g_n^j, j = 1, 2, \ldots J$ and a scaling sequence h_n^J, defined through their z-transforms [7]:

$$H^j(z) = H(z)H(z^2)\dots H(z^{2^j-1})$$
$$G^j(z) = H(z)H(z^2)\dots G(z^{2^j-1})$$

In the following sections, we will construct ON libraries by applying smoothness and similarity criteria to wavelet sequences at the lowest level in the filterbank, g_n^J.

3. PARAMETRIZATION OF ORTHONORMAL WAVELETS

Perfect reconstruction QMF filterbanks were parametrized by Vaidyanathan [9]. Zou and Tewfik [10] imposed the additional condition that the wavelet integrate to zero, thus parametrizing all tight frame (rather than orthonormal) wavelets. This distinction is not crucial, however, because the set of all tight frame wavelets which are not orthonormal is a set with measure zero [4].

Let the filters in the filterbank of Fig. 1 be of length $2(N+1)$ (corresponding to a wavelet of support $[-N, N+1]$). The parametrization of [10] links the z-transforms of the two filters to N paraunitary transfer matrices V_k, $k = 1, \dots, N$:

$$\begin{bmatrix} H(z) \\ z^{-2N}G(z) \end{bmatrix} = \frac{\sqrt{2}}{2} V_N(z)V_{N-1}(z)\cdots V_1(z)V_0 \begin{bmatrix} 1 \\ z^{-1} \end{bmatrix},$$

where $V_k(z) = I + (z^{-2}-1)v_k v_k^T$, and $v_k = \begin{bmatrix}\cos\theta_k & \sin\theta_k\end{bmatrix}^T$ Matrix V_0 is unitary, of the form

$$V_0 = \begin{bmatrix} -\cos\theta_0 & \sin\theta_0 \\ \sin\theta_0 & \cos\theta_0 \end{bmatrix},$$

where angle θ_0 is fixed to be $3\pi/4$ by the condition that the wavelet has vanishing zeroth moment ($\int \psi(t)dt = 0$).

Thus, filters corresponding to ON wavelets of support $[-N, N+1]$ can be generated from N free angles $\theta_1, \dots, \theta_N$, where $0 \leq \theta_k < \pi$.

4. WAVELET SMOOTHNESS

Since the ON parametrization fixes only the zeroth wavelet moment to be zero, the N-dim angle space is composed of both smooth and non-smooth wavelets. The continous-time measure of wavelet smoothness is regularity, which is related to the number of vanishing wavelet moments and can be mathematically expressed through the Holder or Sobolev exponent [6, 7]. Constraints on the angles so that higher moments are equal to zero have been found [10], but this does not necesarily guarantee wavelet smoothness [6, 7]. However, since the basis functions of a practical, finite-level DWT are wavelet sequences $g_n^j, j = 1, \dots J$, a discrete-time measure of smoothness, such as local variation [3, 6], can be used.

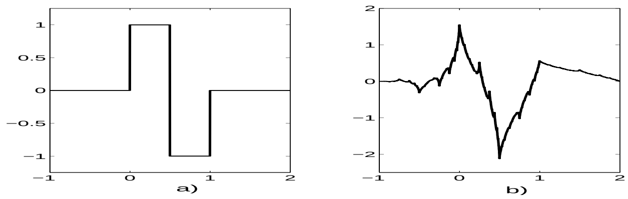

Figure 2: a) The Haar wavelet. b) An equally smooth wavelet ($\theta = .3481$).

The m-th order difference of a wavelet sequence g^J is the sequence $D_n^{(m)}$ [6]:

$$D_n^{(m)} = \sum_{k=0}^{m} (-1)^k \binom{m}{k} g_{n-k}^J.$$

The *local variation* of wavelet sequence is the 1-norm of the m-th order difference : $V^{(m)} = \|D_n^{(m)}\|_1$. The first-order local variation, for example, is simply the sum of the differences between succesive samples: $V^{(1)} = \sum_n |g_n^J - g_{n-1}^J|$. We found that the first order difference was not sensitive enough for our purposes; the examples in the paper were all generated with second order difference $V^{(2)}$.

5. WAVELET SIMILARITY

A measure of wavelet similarity is the discrete inner product between two wavelet sequences $g_{p_n}^J$ and $g_{q_n}^J$:

$$IP(p,q) = 2^{(-J)} \sum_n |g_{p_n}^J g_{q_n}^J| \quad (1)$$

We will consider two wavelet sequences to be similar if their inner product is above some threshold τ_{ip}, where $0 \leq \tau_{ip} \leq 1$. Note that the inner product as defined in (1) can also be computed using the DWT in which one of the wavelet sequences is decomposed using the filters corresponding to the other: the inner product is given by the zeroth coefficient d_0^J of the detail sequence at level J.

6. SMOOTH ORTHONORMAL WAVELET LIBRARIES

Since our aim is to construct a general wavelet library, we will include wavelets that may not be suitable for some applications (those with jumps, for example). The Haar wavelet ($V^{(2)} = 8$) can be used to set the smoothness threshold: in dividing the angle space into smooth and non-smooth regions, we will consider wavelets less smooth than the Haar wavelet to be non-smooth. The Haar wavelet as well as another wavelet of equal smoothness are shown on Fig. 2.

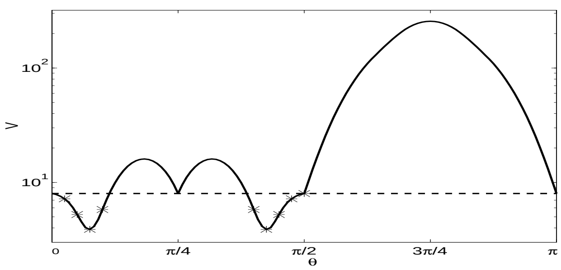

Figure 3: Local variation as a function of angle for $N = 1$. The dotted line is the smoothness threshold. Also shown are the angles of wavelets in a library of size 9.

The dependence of local variation on angle for filters of length 4 (one free angle) is shown on Fig. 3. The dotted line is the smoothness threshold. We can see that more than half of the angle line is not valid. For filters of length 6 ($N = 2$), we show on Fig. 4 the space divided into smooth regions (shaded areas) and the non-smooth regions (the white areas). The smoothness threshold, of course, will differ from aplication to application. However, it is evident that large regions of the angle space are unusable once smoothness is required. The symmetry evident in Fig. 3 and Fig. 4 results from the fact that a wavelet $\psi(t)$ and its "inverse" in time, $\psi(-t)$, have the same local variation.

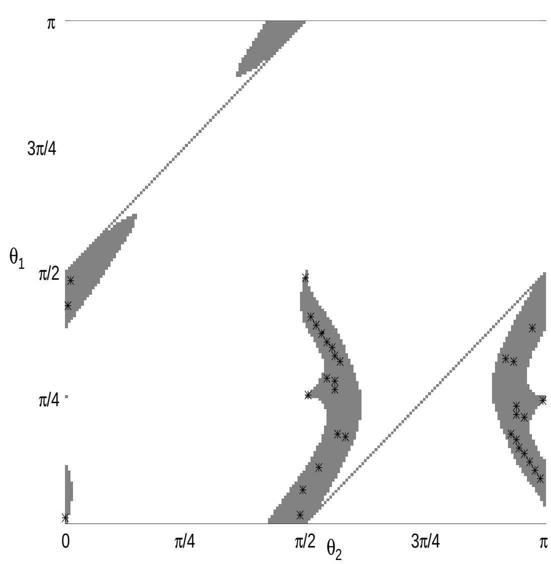

Figure 4: Regions of smoothness (shaded) for $N = 2$. Also shown are the angles corresponding to wavelets in a library of size 34

Once smooth regions in the angle space have been identified, a measure of similarity must be applied to discriminate among remaining wavelet sequences. Small changes in the angle values result in small changes in wavelet shape (for smooth wavelets). Furthermore, the angle parametrization is such that wavelets of similar shapes can be found at widely differing angles. For example, the Haar wavelet (and its translates) can be found at three values for filters of length 4: at $\theta = 0, \pi/4, 3\pi/4$. For $N = 2$, the Haar wavelet can be found not only at several discrete points ($[0, 0], [\frac{\pi}{4}, 0], [\frac{\pi}{4}, \frac{\pi}{2}], [\frac{\pi}{2}, \frac{\pi}{2}]$), but also at all points along the line $|\theta_2 - \theta_1| = \pi/2$ (see Fig. 4).

The similarity threshold determines the library size (the number of distinct wavelets found) and depends on a particular application. There is, of course, a tradeoff between library size and efficiency, so the smallest possible τ_{ip} should be used. Fig. 5 shows the library size L as a function of the inner product threshold for $N = 1, 2, 3$. $J = 6$ levels in the DWT were used to generate the wavelet sequences. We can see that, as expected, for a given τ_{ip}, angle spaces with larger dimensions result in larger library sizes. The actual wavelets in the library are not necessarily unique—if the inner product of two wavelets is larger than τ_{ip}, a choice has to be made which one to eliminate. In the algorithm used here, the smoother of the two wavelets is kept at every step.

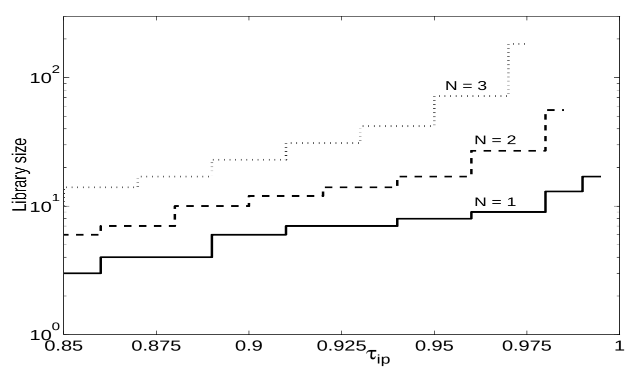

Figure 5: Library size as a function of inner product threshold τ_{ip} for $N = 1, 2, 3$, where N is the number of angles.

Fig. 3 shows the angles corresponding to the 9 wavelets forming the library of filters of length 4, found by setting the smoothness threshold to $\tau_{ip} = .97$. Applying the same threshold to filters of length 6 ($N = 2$), we find 34 wavelets; they are shown on Fig. 4.

7. EXAMPLE APPLICATION

The necessary library size L for a particular application can be determined by comparing compactness in the wavelet domain as τ_{ip} is increased. We will use the entropy of the

DWT coefficients [1] as a measure of compactness in the wavelet domain.

Given an L-level DWT, let $w_k, k = 0, \ldots K$ be the sequence of wavelet coefficients (consisting of the detail coefficients $d_{m,k}$ at all levels and the approximation coefficients at the lowest level $c_{J,k}$), entropy is defined as:

$$E = -\sum_{k=0}^{K} \frac{|w_k|^2}{S} \cdot \log_2 \frac{|w_k|^2}{S}, \qquad (2)$$

where $S = \sum_{k=0}^{K} |w_k|^2$ is the energy in the wavelet coefficients (equal to the energy in the signal sequence). We will normalize entropy by $\log_2 K$ so that its range is $0 \leq E \leq 1$.

Fig. 6a shows one 1024 point window of a noisy 80K point radio frequency interference (RFI) data set. The goal is to isolate the noise-like signal from the RFI [5]; hence the need for smooth wavelets which can be subtracted from the signal while leaving most for the noise intact. We determined the optimum library wavelet for each 1024 point window by minimizing (2) over wavelets of ON libraries of various sizes (for filters of length 6 and 8). The results are given in Fig. 6b, which shows entropy averaged over all the windows of the RFI as a function of library size (for $N = 2, 3$). We can see that for $N = 2$ entropy decreases significantly when the library size is increased from one (just the smoothest wavelet) to about 20 wavelets; after that the decrease in entropy as the library size is increased does not seem significant. Similarly, for $N = 3$, less than 40 wavelets seem to represent the angle space well. The fact that entropy decreases as the number of angles is increased suggests that spaces of higher dimensions will present a better match to this signal set.

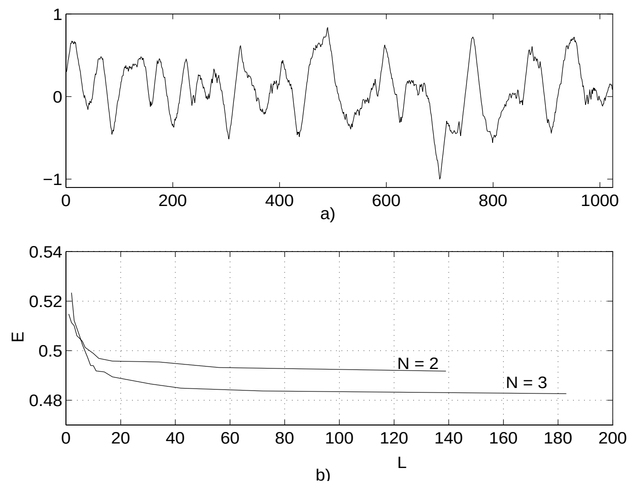

Figure 6: a) One 1024pt window of a noisy 80K RFI data set. b) Entropy averaged over all the windows of the RFI as a function of library size for $N = 2, 3$ (number of angles).

8. SUMMARY

We have constructed smooth ON wavelet libraries using local variation of a wavelet sequence to determine wavelet smoothness and the discrete inner product to discriminate between wavelet shapes for $N = 1, 2, 3$. We applied an entropy-based and an energy-based wavelet selection algorithm to an RFI signal set and compared compactness in the wavelet domain as a function of library size. The results are promising: for the RFI signal set, the number of wavelets needed to represent the N-dim space for $N = 1, 2, 3$ is not large. Further research is needed to extend the results to higher dimensions.

9. REFERENCES

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