A GENERAL APPROACH TO THE GENERATION OF BIORTHOGONAL BASES OF COMPACTLY-SUPPORTED WAVELETS

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ABSTRACT

Biorthogonal bases of compactly-supported wavelets are characterized by the FIR perfect-reconstruction filterbanks to which they correspond. In this paper we develop explicit representations of all such filterbanks, allowing us to generate every possible biorthogonal compactly-supported wavelet basis. For these filterbanks, the product $\mathcal{H}(z) = H(z)\,\widetilde{H}(z)$ of the two lowpass filters must have $N \geq 2$ zeros at $z = -1$. There are $N+1$ minimal-length filterbanks for each N. The filterbanks associated with standard orthogonal and symmetric biorthogonal wavelet bases are found as a special case by using appropriate factorizations of symmetric $\mathcal{H}(z)$ with even N; other filterbanks lead to novel biorthogonal bases.

1. INTRODUCTION

The close relationship between orthonormal wavelet bases and quadrature-mirror filter (QMF) filterbanks is well-known [6, 3, 4]. Daubechies' success in exploiting this relationship led her, along with Cohen and Feauveau, to construct biorthogonal wavelet bases from more general perfect-reconstruction filterbanks [2]. Their work, however, required that the filters be linear phase, and resulted in symmetric wavelets. While this may be desirable in many circumstances, in others it might prove needlessly restrictive. Furthermore, since orthonormal wavelets (other than the Haar wavelet) never exhibit symmetry, Cohen, Daubechies, and Feauveau's construction excludes Daubechies' earlier results for orthonormal wavelet bases. In this paper we generalize their work to formulate methods for generating all FIR perfect-reconstruction filterbanks which lead to biorthogonal bases of regular compactly-supported wavelets (including, as special cases, Daubechies' orthonormal bases and Cohen, Daubechies, and Feauveau's biorthogonal bases).

2. BIORTHOGONAL MULTIRESOLUTION ANALYSIS AND PERFECT-RECONSTRUCTION FILTERBANKS

A biorthogonal multiresolution analysis is specified by dual scaling functions ϕ and $\widetilde{\phi}$ and dual wavelets ψ and $\widetilde{\psi}$. The multiresolution functions satisfy dilation equations:

$$\phi(x) = \sum_n h_n \cdot \sqrt{2}\,\phi(2x-n) \tag{1}$$

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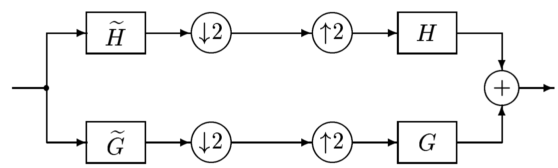

Figure 1: Two-channel perfect-reconstruction filterbank

$$\widetilde{\phi}(x) = \sum_n \widetilde{h}_n^* \cdot \sqrt{2}\,\widetilde{\phi}(2x-n) \tag{2}$$

$$\psi(x) = \sum_n g_n \cdot \sqrt{2}\,\phi(2x-n) \tag{3}$$

$$\widetilde{\psi}(x) = \sum_n \widetilde{g}_n^* \cdot \sqrt{2}\,\widetilde{\phi}(2x-n) \tag{4}$$

(we define $x_n^* = \overline{x}_{-n}$, where the overbar denotes complex conjugation) [2]. The dilation coefficient sequences h, $\widetilde{h}$, g, and $\widetilde{g}$—given by $L^2(\mathbf{R})$ inner products among the multiresolution functions and their dilates—turn out to be impulse responses of the filters of the two-band perfect-reconstruction filterbank shown in Fig. 1; the filters are FIR whenever the multiresolution functions are compactly-supported. Conversely, for appropriate two-channel perfect-reconstruction filterbanks (1)–(4) can be solved for ϕ, $\widetilde{\phi}$, ψ, and $\widetilde{\psi}$, which are compactly-supported if the filters are FIR [4, 5]. Hence finding such filterbanks is the key to constructing biorthogonal bases of compactly-supported wavelets.

3. PERFECT-RECONSTRUCTION CONDITIONS FOR BIORTHOGONAL WAVELETS

Consider the arbitrary two-channel subband coder of Fig. 1 (this is QMF if $\widetilde{h}_n = h_n^*$, $\widetilde{g}_n = g_n^*$, and $g_n = (-1)^n \overline{h}_{1-n}$, but here, in the general case, we presuppose no relationships among the filters). For a filterbank composed entirely of FIR filters, the system will yield perfect reconstruction if and only if

$$H(z)\widetilde{H}(z) + H(-z)\widetilde{H}(-z) = 2, \tag{5}$$

or, in the time domain,

$$\sum_n h_n \widetilde{h}_{2l-n} = \delta_l,$$

where with no real loss of generality we take

$$g_n = (-1)^n \widetilde{h}_{n-1}, \quad \widetilde{g}_n = (-1)^n h_{n+1} \tag{6}$$

[8, 7, 2]. To find solutions to (5), note that $H(z)$ and $\widetilde{H}(z)$ enter into it only through their (FIR) product

$$\mathcal{H}(z) = H(z)\widetilde{H}(z); \tag{7}$$

(5) is really just

$$\mathcal{H}(z) + \mathcal{H}(-z) = 2. \tag{8}$$

Since $\sum_n h_n \widetilde{h}_{2l-n} = (\downarrow 2)(h * \widetilde{h})$, it is clear that all FIR solutions are given by

$$\mathcal{H}(z) = 1 + \sum_{k=k_1}^{k_2} c_{2k-1} z^{-(2k-1)}$$

for arbitrary complex c_{2k-1}. Hence all FIR perfect-reconstruction filterbanks can be obtained by starting with any $\mathcal{H}(z) = \sum_{n=-N_1}^{N_2} c_n z^{-n}$ such that $c_{2k} = \delta_k$ (N_1 and N_2 are thus non-negative and odd if positive), factoring this into FIR $H(z)$ and $\widetilde{H}(z)$, and then constructing g and $\widetilde{g}$ from h and $\widetilde{h}$ according to (6).

While this provides a general characterization of FIR perfect-reconstruction filterbanks, for the filterbanks to give rise to biorthogonal wavelet bases we must impose further constraints on H and $\widetilde{H}$ [2]. First, both $H(1)$ and $\widetilde{H}(1)$ must be $\sqrt{2}$, or equivalently $\mathcal{H}(1) = 2$ (with the understanding that the factorization is normalized appropriately). Furthermore, the filters H and $\widetilde{H}$ must have at least one zero at $z = -1$ each, i.e., $\mathcal{H}(z)$ must have $N \geq 2$ zeros at $z = -1$; in fact, for regular (smooth) scaling functions and wavelets, the filters must have even more zeros there [5]. Conveniently, with $\mathcal{H}(-1) = 0$ (8) implies automatically that $\mathcal{H}(1) = 2$, taking care of the first requirement. The second means that

$$\mathcal{H}(z) = (z+1)^N P(z), \tag{9}$$

where $N \geq 2$ and $P(z)$ is a Laurent polynomial (a polynomial with potentially positive *and* negative exponents). Combining (8) and (9), then, the central problem in finding FIR perfect-reconstruction filterbanks which yield regular biorthogonal wavelet bases is determining Laurent polynomial solutions $P(z)$ to

$$A(z)\,P(z) + A(-z)\,P(-z) = 2 \tag{10}$$

for $A(z) = (z+1)^N$ and $N \geq 2$.

4. SOLVING THE WAVELET PERFECT RECONSTRUCTION EQUATION

Assume for the moment that (10) has some solution, say $P_0(z)$ (we will see shortly that this is so). Then there are actually infinitely many more, given by

$$P(z) = P_0(z) + R(z)\,A(-z), \tag{11}$$

where $R(z)$ is any Laurent polynomial satisfying $R(-z) = -R(z)$ (this is evident by direct substitution of (11) into the left-hand side of (10)). In fact, (11) encompasses *all* solutions. For let $P_1(z)$ and $P_2(z)$ be two Laurent solutions to (10), and call their (Laurent) difference $P'(z) = P_1(z) - P_2(z)$. Then

$$\begin{aligned} 0 &= \left[A(z)\,P_1(z) + A(-z)\,P_1(-z)\right] \\ &\qquad - \left[A(z)\,P_2(z) + A(-z)\,P_2(-z)\right] \\ &= A(z)\,P'(z) + A(-z)\,P'(-z); \end{aligned}$$

this shows that $A(z)\,P'(z) = -A(-z)\,P'(-z)$ and hence that $A(-z) = (1-z)^N$ can be factored from $A(z)\,P'(z)$. Of course the zeros of $A(z)$ are at $z = -1$, not $z = 1$, so $A(-z)$ must be a factor of $P'(z)$: $P'(z) = A(-z)\,R(z)$. Furthermore, $A(z) \cdot A(-z)\,R(z) = -A(-z) \cdot A(z)\,R(-z)$, or $A(z)\,A(-z)\big[R(z) + R(-z)\big] = 0$. This can happen only if $R(z) + R(-z)$ is identically 0, i.e., only if $R(-z) = -R(z)$. Necessarily, then, any two solutions of (10) differ by the product of $A(-z)$ and an odd Laurent polynomial. Thus (11) provides all solutions to (10) should a particular one exist.

To show the existence of particular solutions, we use Bezout's theorem [2], a classical result for (ordinary, not Laurent) polynomials. It says that given two polynomials $a(x)$ and $b(x)$ of degrees N_a and N_b, respectively, then if $a(x)$ and $b(x)$ have no common zeros there exist unique polynomials $p(x)$ and $q(x)$ of degrees at most $N_b - 1$, $N_a - 1$, respectively, such that

$$a(x)\,p(x) + b(x)\,q(x) = 1.$$

Surprisingly enough, this is equivalent to a more general result, namely, that under the same circumstances, for *any* polynomial $c(x)$ of degree $N_a + N_b - 1$ or less there exist unique $p(x)$ and $q(x)$ (satisfying the same degree conditions) such that

$$a(x)\,p(x) + b(x)\,q(x) = c(x).$$

One way to see this is to write out the left-hand side in terms of the coefficients of a, b, p, and q and match the result to the right-hand side, which yields a system of $N_a + N_b$ linear equations in as many unknowns. The standard Bezout theorem guarantees a unique solution for one particular right-hand side; for square linear systems, though, this implies the existence of a unique solution for any right-hand side.

To apply Bezout's theorem to (10), recall from section 3 that $\mathcal{H}(z) = c_{-N_1} z^{N_1} + \ldots + c_{N_2} z^{-N_2}$ for N_1 and N_2 that are either 0 or else odd and positive. Comparing this with (9) shows that the terms of $P(z)$ have exponents ranging from $N_1 - N$ down to $-N_2$. So if $P_0(z)$ is a particular solution to (10), we can write

$$P_0(z) = z^{-N_2} p_0(z), \tag{12}$$

where $p_0(z)$ is an ordinary polynomial satisfying

$$A(z)\,p_0(z) + (-1)^{N_2} A(-z)\,p_0(-z) = 2z^{N_2}. \tag{13}$$

Since $A(z) = (z+1)^N$ shares no roots with $(-1)^{N_2} A(-z)$, the generalized version of Bezout's theorem tells us that for $N_2 = 0, 1, 3, \ldots, 2N-1$ there are unique ordinary $p(z)$ and $q(z)$ of degree $N-1$ or less such that

$$A(z)\,p(z) + (-1)^{N_2} A(-z)\,q(z) = 2z^{N_2}.$$

If we substitute $-z$ for z, this becomes

$$A(-z)\,p(-z) + (-1)^{N_2} A(z)\,q(-z) = 2(-1)^{N_2} z^{N_2},$$

or

$$A(z)\,q(-z) + (-1)^{N_2} A(-z)\,p(-z) = 2z^{N_2}.$$

But notice that $\deg\big[q(-z)\big] = \deg q \le N-1$ and likewise $\deg\big[p(-z)\big] \le N-1$. So actually, by the uniqueness of the Bezout solution, $q(z) = p(-z)$. Thus for each $N_2 = 0, 1, 3, \ldots, 2N-1$ there is in fact an ordinary polynomial $p_0(z)$ satisfying (13), which along with (12) establishes the existence of particular solutions to (10).

While there are implicit methods for finding Bezout solutions, for (13), where $A(z) = (z+1)^N$, we can use reasoning similar to that in [2] to determine $p_0(z)$ explicitly. With the transformation $w = 1 - z$ (so that $z = 1 - w$), (13) becomes

$$(2-w)^N p_0(1-w) + (-1)^{N_2} w^N p_0(w-1) = 2(1-w)^{N_2},$$

or

$$p_0(1-w) = 2(1-w)^{N_2}(2-w)^{-N} - (-1)^{N_2} w^N (2-w)^{-N} p_0(w-1). \tag{14}$$

Now do a Taylor expansion of $(2-w)^{-N}$:

$$(2-w)^{-N} = \left(\tfrac{1}{2}\right)^N \sum_{k=0}^{\infty} \left(\tfrac{1}{2}\right)^k \binom{N+k-1}{k} w^k.$$

Since $(1-w)^{N_2} = \sum_{k=0}^{N_2} (-1)^k \binom{N_2}{k} w^k$ and the coefficients of the product of polynomials is the convolution of their coefficients,

$$2(1-w)^{N_2}(2-w)^{-N} = \left(\tfrac{1}{2}\right)^{N-1} \sum_{k=0}^{\infty} \Bigg[\sum_{m=\max(k-N_2,0)}^{k} (-1)^{k-m} \binom{N_2}{k-m} \cdot \left(\tfrac{1}{2}\right)^m \binom{N+m-1}{m} \Bigg] w^k.$$

Notice, though, that $p_0(w)$ is a polynomial of degree $N-1$ or less, and hence that $p_0(1-w)$ and $p_0(w-1)$ are as well. So the left-hand side of (14) has no terms of degree N or higher. The second part of the right-hand side of (14), however, has *only* terms of degree N or more. These, then, must cancel the corresponding high-degree terms of the first part of the right-hand side, and thus

$$p_0(1-w) = \left(\tfrac{1}{2}\right)^{N-1} \sum_{k=0}^{N-1} \Bigg[\sum_{m=\max(k-N_2,0)}^{k} (-1)^{k-m} \left(\tfrac{1}{2}\right)^m \cdot \binom{N+m-1}{m} \binom{N_2}{k-m} \Bigg] w^k.$$

Putting this in terms of z, expanding $(1-z)^k$, and using (12), we get for $N_2 = 0, 1, 3, \ldots, 2N-1$ the particular solutions

$$P_0(z) = \left(\tfrac{1}{2}\right)^{N-1} z^{-N_2} \cdot \sum_{n=0}^{N-1} (-1)^n \Bigg[\sum_{k=n}^{N-1} \binom{k}{n} \cdot \sum_{m=\max(k-N_2,0)}^{k} (-1)^{k-m} \left(\tfrac{1}{2}\right)^m \cdot \binom{N+m-1}{m} \binom{N_2}{k-m} \Bigg] z^n. \tag{15}$$

5. EXAMPLES

From (11) it is obvious that for any N there are arbitrarily long $P(z)$ which satisfy (10). The particular solutions given by (15), on the other hand, are short. For example, with $N = 2$, we have

$$\begin{aligned} P_0(z) &= -\tfrac{1}{2}z + 1 && (N_2 = 0) \\ P_0(z) &= \tfrac{1}{2}z^{-1} && (N_2 = 1) \\ P_0(z) &= z^{-2} - \tfrac{1}{2}z^{-3} && (N_2 = 3); \end{aligned}$$

with $N = 3$,

$$\begin{aligned} P_0(z) &= \tfrac{3}{8}z^2 - \tfrac{9}{8}z + 1 && (N_2 = 0) \\ P_0(z) &= -\tfrac{1}{8} + \tfrac{3}{8}z^{-1} && (N_2 = 1) \\ P_0(z) &= \tfrac{3}{8}z^{-2} - \tfrac{1}{8}z^{-3} && (N_2 = 3) \\ P_0(z) &= z^{-3} - \tfrac{9}{8}z^{-4} + \tfrac{3}{8}z^{-5} && (N_2 = 5). \end{aligned}$$

Notice that when $N_2 = 0$ or $2N-1$ the $P_0(z)$ have length N, while for intermediate values of N_2 they have length $N-1$. This holds in general: for $N_2 = 0$ we see that $P_0(0) = 1$ from (12) and (13) and that the coefficient of z^{N-1} is non-zero from (15). From this, (11), the fact that N_1 and N_2 must be odd and positive or else zero, and the uniqueness of Bezout solutions one can see that the lengths of the $P_0(z)$ must follow the pattern observed above. Furthermore, (11) shows that all other solutions are longer.

Look again at the $N = 2$ solutions. When $N_2 = 1$, $P_0(z)$ leads to Haar wavelets. In fact, since the derivation assumed nothing other than what is strictly necessary for the resulting filterbanks to correspond to biorthogonal wavelet bases, we fully expect to find other families of orthogonal and biorthogonal wavelets as particular solution cases. For example, with $N = 4$ and $N_2 = 3$,

$$P_0(z) = -\tfrac{1}{16}z^{-1} + \tfrac{1}{4}z^{-2} - \tfrac{1}{16}z^{-3}.$$

This has two roots, $z = 0.2679$ and $z = 3.7321$. Assigning the smaller to H along with two zeros at $z = -1$ and using the larger and the other two $z = -1$ zeros for $\widetilde{H}$ produces the classic Daubechies $N = 2$ orthonormal functions (our N is twice that in [3]). Alternatively, keeping $P_0(z)$ intact and pairing it with some of the zeros at $z = -1$ yields symmetric biorthogonal bases as in [2]. In general, we find both Daubechies' orthonormal and Cohen, Daubechies, and Feauveau's biorthogonal bases from appropriate factorizations of solutions with even N and $N_2 = N - 1$.

For other factorizations, other N_2, or odd N, we get filterbanks which correspond to novel biorthogonal bases (even complex-valued ones). As a fully-worked example of this situation, consider $N = 5$ and $N_2 = 7$. For these parameters

$$P_0(z) = \tfrac{35}{128}z^{-4} - \tfrac{47}{128}z^{-5} + \tfrac{25}{128}z^{-6} - \tfrac{5}{128}z^{-7},$$

with roots at $z = 0.4366 \pm 0.3370i$ and $z = 0.4696$. Pairing the two complex roots with 3 $z = -1$ zeros for H and the real root with the remaining 2 $z = -1$ zeros for $\widetilde{H}$ gives

$$H(z) = 0.4102 + 0.8724z^{-1} + 0.2808z^{-2} - 0.2901z^{-3} + 0.01614z^{-4} + 0.1248z^{-5},$$

$$\widetilde{H}(z) = 0.6666z + 1.02 + 0.0405z^{-1} - 0.3131z^{-2}$$

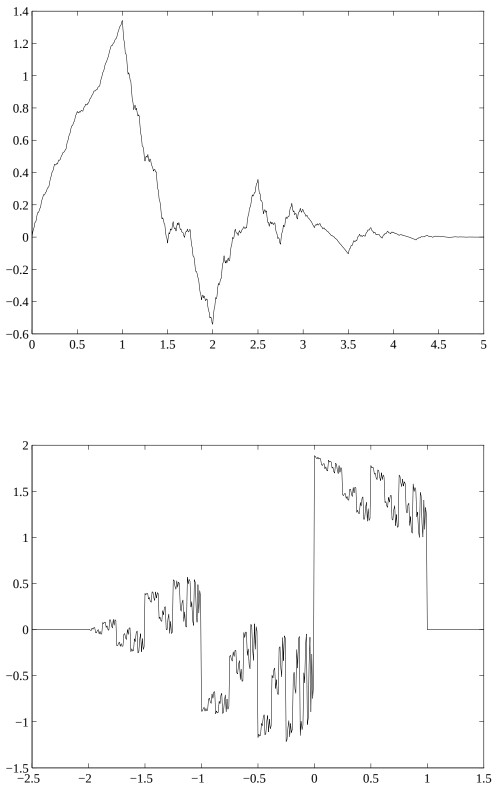

Figure 2: Scaling function ϕ (top) and dual $\widetilde{\phi}$ (bottom)

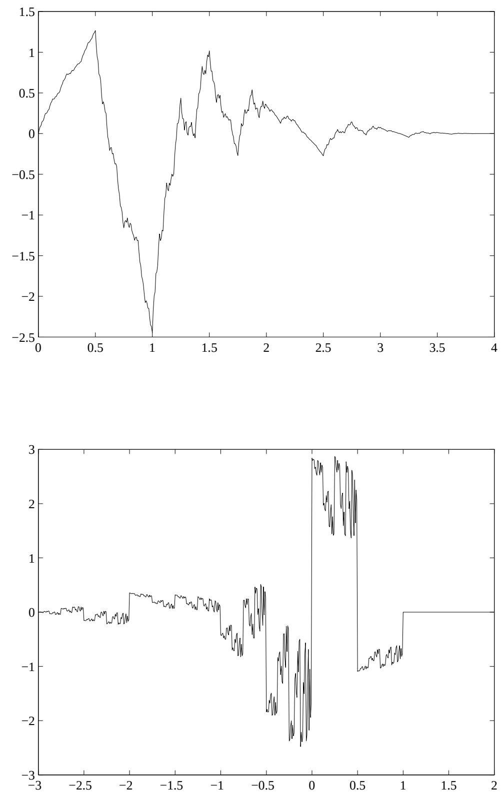

Figure 3: Wavelet ψ (top) and dual $\widetilde{\psi}$ (bottom)

(note that $H(1) = \widetilde{H}(1) = \sqrt{2}$). Finally, obtaining g and $\widetilde{g}$ from (6), we get biorthogonal multiresolution analysis functions from (1)–(4). The dual scaling functions ϕ and $\widetilde{\phi}$ are shown in Fig. 2; the dual wavelets ψ and $\widetilde{\psi}$ appear in Fig. 3. It should be noted that this procedure doesn't always succeed; some of the filterbanks created this way do not actually correspond to a valid biorthogonal multiresolution analysis. There is, however, an efficient and definitive test that can be applied to $H(z)$ and $\widetilde{H}(z)$ that will verify if the filterbank is associated with a biorthogonal wavelet basis [1] (as is the case, for instance, in our example).

6. CONCLUSIONS

We have presented an explicit characterization of all FIR perfect-reconstruction filterbanks for which the product filter $\mathcal{H}(z) = H(z)\widetilde{H}(z)$ has 2 or more zeros at $z = -1$. Since these are precisely the filterbanks which give rise to biorthogonal bases of regular compactly-supported wavelets, then, we have an effective general framework for generating every possible such basis, free of any particular restrictions such as orthogonality or wavelet symmetry.

7. REFERENCES

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