EQUALIZATION AND LINEARIZATION OF NONLINEAR SYSTEMS

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ABSTRACT

This paper presents a theory for the exact and the pth order equalization or linearization of nonlinear systems with known recursive or nonrecursive polynomial input-output relationships. The equalizing and linearizing filters have simple and computationally efficient structures. An experimental result that illustrates the good properties of the technique we propose is also included in this paper.

1. INTRODUCTION

Equalization of linear systems has been studied for several years. Many real channels, however, possess non-negligible nonlinearities that make it impossible for linear equalization procedures to provide acceptable results. Examples of real world systems in which nonlinear effects are present include satellite communication channels, voiceband data transmission systems, high density magnetic recordings, high density optical systems and loudspeaker systems. This paper presents a theory for the exact and the pth order equalization or linearization of nonlinear systems with known polynomial input-output relationships.

Definition 1 *A* nonlinear equalizer *is a filter which, when connected in cascade before or after a nonlinear system, results in an overall system whose characteristics corresponds to those of an identity system in the band of frequencies and in the range of input signal amplitudes of interest.*

Definition 2 *A* linearizer *is a filter which, when connected in cascade before or after the unknown system, results in an overall system whose characteristics correspond to those of a linear system in the frequency band and in the range of input signal amplitudes of interest.*

When an equalizer (linearizer) is connected before a nonlinear system, it is called a *pre-equalizer* (*pre-linearizer*). When it is connected after a nonlinear system, it is called a *post-equalizer* (*post-linearizer*).

Many equalization/linearization procedures are available in the literature [1, 2, 4, 5]. Many such techniques identify an approximate, truncated Volterra system to perform the equalization. One exception to this framework is [5], in which a method for the blind equalization of truncated Volterra channels by means of a bank of linear filters is presented. Even though we do not consider blind equalization in this paper, our approach is capable of exact pre- and post- equalization or linearization of nonlinear channels. We note that the method in [5] cannot be used for pre-equalization of nonlinear systems. In addition, our method is useful for a much larger class of channel models.

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2. IDEAL EQUALIZATION/LINEARIZATION

We consider the following system model:

$$y(n) = \sum_{i=0}^{N} a_i x(n-i) + \sum_{k=2}^{L} \sum_{i_1=1}^{N} \sum_{i_2=i_1}^{N} \cdots \sum_{i_k=i_{k-1}}^{N} h_{i_1 i_2 \ldots i_k} x(n-i_1) x(n-i_2) \cdot \ldots \cdot x(n-i_k). \quad (1)$$

The theory we present is applicable to a larger class of nonlinear systems including recursive polynomial models. However, we restrict the discussions to the above model for ease of presentation. The inverse of the system in (1) is given by [3]

$$w(n) = \frac{1}{a_0}\left[u(n) - \sum_{i=1}^{N} a_i w(n-i) - \sum_{k=2}^{L} \sum_{i_1=1}^{N} \cdots \sum_{i_k=i_{k-1}}^{N} h_{i_1 i_2 \ldots i_k} w(n-i_1) w(n-i_2) \cdot \ldots \cdot w(n-i_k)\right]. \quad (2)$$

An explicit expression for the output of the inverse system as above is possible because the system model does not depend on the input sample $x(n)$ in a nonlinear manner. An implicit expression for the inverse of the more general system

$$y(n) = \sum_{i=0}^{N} a_i x(n-i) + \sum_{k=2}^{L} \sum_{i_1=0}^{N} \sum_{i_2=i_1}^{N} \cdots \sum_{i_k=i_{k-1}}^{N} h_{i_1 i_2 \ldots i_k} x(n-i_1) x(n-i_2) \cdot \ldots \cdot x(n-i_k), \quad (3)$$

if it exists, is given by

$$w(n) = \frac{1}{a_0}\left[u(n) - \sum_{i=1}^{N} a_i w(n-i) - \sum_{k=2}^{L} \sum_{i_1=0}^{N} \cdots \sum_{i_k=i_{k-1}}^{N} h_{i_1 i_2 \ldots i_k} w(n-i_1) w(n-i_2) \cdot \ldots \cdot w(n-i_k)\right]. \quad (4)$$

In what follows, we employ the compact expression

$$y(n) = A(q)x(n) + N\big[x(n)\big] \quad (5)$$

to represent the systems in (1) and (3). In the above equation, q^{-1} is the delay operator, $x(n)$ and $y(n)$ are the input and output signals, respectively,

$$A(q) = \sum_{i=0}^{N} a_i q^{-i} \tag{6}$$

and

$$N\big[x(n)\big] = \sum_{k=2}^{L} \sum_{i_1=r}^{N} \cdots \sum_{i_k=i_{k-1}}^{N} h_{i_1 \ldots i_k} x(n-i_1) \cdot \ldots \cdot x(n-i_k) \tag{7}$$

If $r = 1$, (5) corresponds to the system in (1), while if $r = 0$, (5) corresponds to the system in (3). Using the above notation, we can express the input-output relationship of the inverse of the system in (5) as

$$A(q)w(n) = u(n) - N\big[w(n)\big], \tag{8}$$

where $u(n)$ and $w(n)$ are the input and output signals, respectively of the inverse system. We can also express the output signal explicitly as

$$w(n) = A^{-1}(q)u(n) - A^{-1}(q)N\big[w(n)\big]. \tag{9}$$

Figure 1 shows a block diagram for the recursive polynomial filter in (9). We note that the overall system can be described as a feedback system in which the feedback loop contains a nonlinear operator and the feedforward loop contains the inverse of the linear component of the nonlinear filter. The inverse system in (5) will not be stable unless

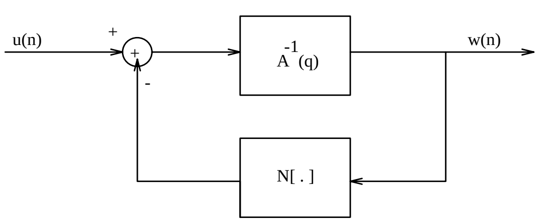

Figure 1: The ideal equalizer.

the linear part of the unknown system model has minimum phase characteristics. When $A(q)$ represents a minimum phase system, the system in (9) can be shown to operate in a stable manner whenever the input signal is sufficiently small. The bound on the input signal depends on the zeros of $A(q)$ and on the coefficients of $N[u(n)]$ [7]. Thus, our inverse system will equalize the above nonlinear system only in the range of amplitudes for which it is stable.

In many applications, we are interested in equalizing the unknown system only in a certain band of frequencies. For example it may be known that the input signal is band-limited. Such an equalizer may be designed by replacing A^{-1} in Figure 1 with another linear filter such that the overall system response corresponds to that of a linear system with specified amplitude and phase response and possibly zero response outside the band of interest. The following two theorems characterize the structure of the post and pre-equalizers for a specific range of frequencies.

Theorem 1 *If the input signal is band-limited with spectrum inside a certain band B, a post-equalizer in the band B for the system of (5) is given by*

$$w(n) = \tilde{A}^{-1}(q)u(n) - \tilde{A}^{-1}(q)N\big[w(n)\big], \tag{10}$$

where $\tilde{A}^{-1}(q)$ is the linear equalizer of the system $A(q)$ in the band B and has zero response outside the band.

Proof: We consider the post-equalization of the nonlinear system in (5) in the band B and the elimination of all other frequencies at the output. For this purpose, we first cascade the system in (5) with the linear filter $\tilde{A}^{-1}(q)$ and then equalize the resulting nonlinear system. Cascading (5) with the linear system $\tilde{A}^{-1}(q)$ eliminates all frequencies outside the band B. The resulting system has input-output relationship

$$z(n) = \tilde{A}^{-1}(q)A(q)x(n) + \tilde{A}^{-1}(q)N[x(n)]. \tag{11}$$

Since $x(n)$ has frequency components only on B, the above system is equivalent to

$$z(n) = x(n) + \tilde{A}^{-1}(q)N[x(n)], \tag{12}$$

whose post-inverse system is given by

$$w(n) = z(n) - \tilde{A}^{-1}(q)N[w(n)]. \tag{13}$$

Since $z(n)$ is band-limited to the band B, the output of the system in (13) is also band-limited to the band B. Thus, the cascade of $z(n) = \tilde{A}^{-1}(q)u(n)$ (where we assume $u(n) = y(n)$) and the system in (13) is the ideal equalizer for the system in (5) in the band B. It is trivial to prove that this system is the same of equation (10). □

Theorem 2 *A pre-equalizer in the band B for the system of (5) is given by*

$$w(n) = \tilde{A}^{-1}(q)u(n) - \tilde{A}^{-1}(q)N\big[w(n)\big], \tag{14}$$

where $\tilde{A}^{-1}(q)$ is the linear equalizer of the system $A(q)$ in the band B and has zero response outside the band.

The proof is similar to that for Theorem 1. For the case of the pre-equalizer, we do not require that the input signal is band-limited. However, we are now unable to compensate for the frequency components of $y(n)$ that fall outside the band B. We also note that the equalizer in (10) and (14) is realizable for the system in (1), but not for the system in (3).

2.1. The ideal pre- and post-linearizers

In this paper, the linear system that results from the linearization process will always have transfer function equal to the linear part of the nonlinear system that is linearized. The *ideal* pre-linearizer (post-linearizer) is the filter that pre-linearizes (post-linearizes) the nonlinear system in all the frequency domain.

The ideal pre-linearizing filter for the system in (5) is given by

$$w(n) = v(n) - A^{-1}(q)N\big[w(n)\big], \tag{15}$$

where $v(n)$ and $w(n)$ are the input and the output signals, respectively.

Proof: By substituting the input-output relationship $u(n) = A(q)v(n)$ in (7) we obtain

$$A(q)w(n) = A(q)v(n) - N\big[w(n)\big], \qquad (16)$$

which is the same system of (15). □

In a similar manner, we can prove that the ideal post-linearizing filter for (5) is given by

$$w(n) = v(n) - N\big[A^{-1}(q)w(n)\big]. \qquad (17)$$

Figures 2 illustrates the block diagram of the ideal pre-linearizer. A similar block diagram can be drawn for the ideal post-linearizer. The following results can be proved in

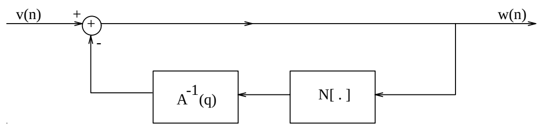

Figure 2: The ideal pre-linearizer.

a manner similar to that employed for proving Theorem 1.

Theorem 3 *If the input signal of the system (5) is band-limited with spectrum inside a certain band B and if $\tilde{A}^{-1}(q)$ is the linear equalizer of the system $A(q)$ in the band B and has zero response outside this band, the post-linearizing filter in the band B for the system of (5) is given by the input-output relationship*

$$w(n) = v(n) - \tilde{A}^{-1}(q)N\big[w(n)\big]. \qquad (18)$$

Theorem 4 *If $\tilde{A}^{-1}(q)$ is is as described in Theorem 3, the pre-linearizing filter in the band B for the system in (5) is given by the input-output relationship*

$$w(n) = v(n) - N\big[\tilde{A}^{-1}(q)w(n)\big]. \qquad (19)$$

3. pTH ORDER EQUALIZATION

The ideal equalizers and linearizers of the previous section may not always be realizable. For example when $r = 0$ in (5), the equalizers and linearizers do not have an explicit input-output relationship. Furthermore, because of the recursive structure of the equalizers/linearizers, these filters may also be unstable. In what follows, we present a theory for the pth order equalization/linearization of nonlinear systems [9].

Definition 3 *A* pth order equalizer *is a filter which, when connected in cascade before or after a nonlinear system, results in an overall system whose characteristics, in the band of frequencies and in the range of input signal amplitudes of interest, corresponds to those of a parallel connection of an identity system and a nonlinear component whose Volterra kernels of order smaller than or equal to p are all zero.*

Definition 4 *A* pth order linearizer *is a filter which, when connected in cascade before or after a nonlinear system, results in an overall system whose characteristics, in the band of frequencies and in the range of input signal amplitudes of interest, correspond to those of a parallel connection of a linear system and a nonlinear component whose Volterra kernels of order smaller than or equal to p are all zero.*

Theorem 5 *If the system of (8) is stable, the sequence of systems defined by*

$$\begin{aligned} w_1(n) &= A^{-1}(q)u(n), & (20)\\ w_p(n) &= A^{-1}(q)u(n) - A^{-1}(q)N\big[w_{p-1}(n)\big] & (21)\end{aligned}$$

converge to the system in (8) when p tends to infinity. Moreover, the system in (21) is a generalized pth order inverse of the system in (5) in the sense of Sarti and Pupolin [8].

Proof: We prove by induction that (21) is the p-th order inverse of (5). Let us process the output of the system of (5) with the system defined by (21). By considering $u(n) = y(n)$, for $p = 1$ we obtain the following input-output relationship:

$$\begin{aligned} w_1(n) &= x(n) + A^{-1}(q)N\big[x(n)\big], \\ &= x(n) + T_1(n), & (22)\end{aligned}$$

where $T_1(n)$ is a Volterra operator of order greater than 1. This proves that $w_1(n)$ is the output of a first order inverse in the sense of Sarti and Pupolin. Let us suppose that $w_p(n)$ is the output of a pth order inverse, *i.e.*,

$$w_p(n) = x(n) + T_p(n) \qquad (23)$$

where $T_p(n)$ is a Volterra operator of order greater than p. We want to prove that the system defined by

$$w_{p+1}(n) = A^{-1}(q)u(n) - A^{-1}(q)N\big[w_p(n)\big] \qquad (24)$$

is a $(p+1)$th order inverse of (5). By substituting (5) and (23) in (24), we have

$$\begin{aligned} w_{p+1}(n) &= x(n) + A^{-1}(q)N\big[x(n)\big] \\ &\quad - A^{-1}(q)N\big[x(n) + T_p(n)\big] \\ &= x(n) + A^{-1}(q)N\big[x(n)\big] \\ &\quad - A^{-1}(q)N\big[x(n)\big] + T_{p+1}(n), & (25)\end{aligned}$$

where T_{p+1} is an operator of order greater than $p+1$ and we have taken into account the fact that $N[\cdot]$ is an operator of order greater than 1. Since the system in (8) is the inverse of (5) and we have shown that the sequence of systems defined by (20)-(21) define pth order inverses of (5), the sequence of systems (20)-(21) will converge to (8) if the system (8) is stable. □

The sequence (20)-(21) corresponds to a systolic cascade of cells. Thus, the pth order equalizer can be easily implemented using VLSI circuits. Furthermore, each cell is always realizable while we recall that the ideal equalizer is not realizable for the system in (3). The block diagram of a pre-equalizer cell is shown in Figure 4. Extensions of the pth order equalizer/linearizer to the cases considered in Theorems 1–4 are possible. Due to space limitations, such a discussion is not included here.

4. AN EXPERIMENTAL RESULT

In this example, we consider the linearization of the nonlinearities associated with a synthetic loudspeaker using the pth order linearizer of Figure 4. Previous work [4] has shown that loudspeaker nonlinearities can be efficiently modelled

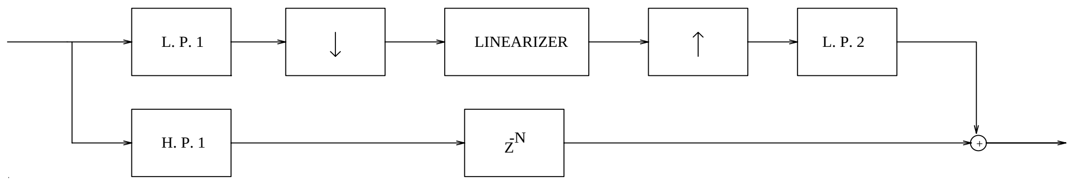

Figure 3: The subband linearizer.

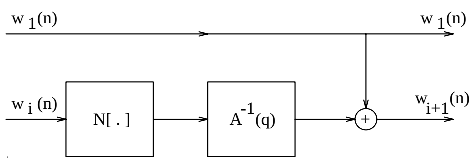

Figure 4: The ith order cell.

with good accuracy using low-order, truncated Volterra systems. The loudspeaker in our experiment was modelled using a quadratic filter with one hundred linear coefficients and a forty-sample memory for the second-order component. The second-order harmonic distortion of the system is shown in Figure 5. Since the distortions were primarily in the range $\left[0, f_N/3\right]$ Hz., where f_N denotes the Nyquist frequency, we employed the system of Figure 3 to perform a pth order linearization. The upper branch of the system contained a PCAS lowpass filter [6] with cut-off frequency $f_N/3$. The output of the lowpass filter was subsampled by a factor three. The parameters of the loudspeaker were estimated using a quadratic filter with 51-sample memory length for the linear component and 40-sample memory for the second-order nonlinearity from subsampled versions of the input to the loudspeaker and its output in the presence of uncorrelated, 30 dB measurement noise. The estimated model was then used to pre-linearize the system using second, third, fourth and fifth order linearizers.

Figure 5 also shows the second-order harmonic distortion measured at the output of the linearized systems. We note that the second-order distortion is the smallest in the case of the second-order linearizer. This linearizer is sufficient to correct for the second-order distortions and it produces the most compact spectrum for the predistorted signal. The third-order linearizer exhibits a higher second-order distortion in this experiment. This is due to model mismatch, our approximations and the wider band of the predistorted signal whose intermodulation contributions alter the amplitude of the fundamental frequency components. The higher-order linearizers exhibit comparable second-order distortions to the second-order linearizer. The improvement due to the use of the linearizer is evident from all our experiments.

5. CONCLUDING REMARKS

This paper presented a theory for the exact and the pth order equalization or linearization of nonlinear systems with known polynomial input-output relationships. An attractive aspect of the results in the paper is that the equalizers and linearizers can be implemented cascading modular and stable components. Experiments indicate that the method works well in situations where the parameters of the nonlinear system must be estimated.

6. REFERENCES

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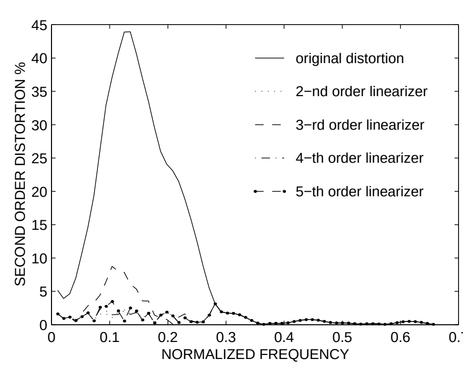

Figure 5: Second-order distortion measured in the experiments.