ERROR REDUCTION OF RANGE ESTIMATES IN MULTIPATH ENVIRONMENTS

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ABSTRACT

A reduction in error is obtained for estimates of range computed in a multipath environment. A good estimation technique for this problem typically involves exploiting the multipath interference, that is, analyzing each received signal reflection to improve the estimate. Under certain conditions, however, the errors in multipath estimates can be quite large. We compute the Cramér-Rao bounds for an example satisfying these conditions to demonstrate this phenomenon. We propose a modification to the original signal model which better represents the received signals. While the modified model does introduce a bias error, we provide a suitable estimate of the bias error so that a complete error anaylsis of the modified model is possible. The total error (noise error plus bias error) for the modified model is computed for an example and compares favorably with the error obtained via the original signal model.

1. INTRODUCTION

For problems involving parameter estimation, the error performance of a processing method can be characterized by the Cramér-Rao bounds. These bounds provide a lower limit on the error under certain conditions. By computing these bounds, it is straightforward to establish the suitability of a design. If the performance of a method approaches this limit, then the method is deemed satisfactory. A recognized phenomenon which complicates the use of Cramér-Rao bounds is the problem of resolvability of a signal model, [1]. Under conditions when components of a signal are similar, a model under study can yield lower bounds which are quite large. Such bounds would suggest that no accurate estimate of the desired parameter(s) could be obtained with that model.

Such phenomenon can be found in the context of range estimation problems in radar when multipath interference occurs. Multipath refers to the situation where an antenna array receives not only the original source signal(s), but replicas of the original source signal(s). These replicas appear because of signal reflections. The bounds for range estimates in multipath environments have been studied; see, for example, [2, 3, 4]. A related formulation for the general parameter estimation problem was given in [5].

Cramér-Rao bounds are typically computed using a few fundamental assumptions. One is that the noise present in a signal is zero-mean white Gaussian. Another is that the estimates being obtained are unbiased, although there is a straightforward modification of the bounds when there is bias present, [6]. Using a simple model loosely based on that in [2] involving one source and one reflection, we will illustrate circumstances where the Cramér-Rao bounds yield unacceptably high bounds. By allowing a small bias, we formulate a modication to the estimation procedure and demonstrate its improved error performance when the signal model is otherwise unresolvable.

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2. PROBLEM DEFINITION

Consider the case of a source with known (or otherwise estimated) bearing at an unknown distance from an antenna array. A realistic example would be an aircraft carrier landing problem, where the distance between the plane and the ship is determined using the radar of the aircraft. The source signal has a known spectrum, and the array receives the original signal plus a reflection signal. In the landing problem, this second received signal could correspond to the reflection of the source off a reflector. Figure 1 illustrates the geometry involved in this problem.

Let α be the range which is to be estimated. As in [2], the signal arriving at receiver k at time t is given by

$$x_m(t) = \sum_{k=1}^{2} a_{mk} s(t - \tau_{mk}(\alpha)) + n_m(t), \quad m = 1, \ldots, M, \tag{1}$$

where $s(t)$ is the signal emitted by the source, a_{mk} is the attenuation of the source to receiver m via path k, $\tau_{mk}(\alpha)$ is the delay from the source to receiver m via path k, and $n_m(t)$ is the noise observed at the array, assumed to be zero-mean white Gaussian and uncorrelated with the source. We assume that the attenuation parameters, a_{mk}, do not depend on the range α. We also assume we know the dependence of the delays τ_{mk} on α. Using the large time-bandwidth assumption, the Fourier components of the received signal $x(t)$ are uncorrelated and can be expressed for each frequency ω as

$$x_m(\omega) = \sum_{k=1}^{2} a_{mk} e^{j\omega\tau_{mk}} s(\omega) + n_m(\omega). \tag{2}$$

We further assume no dependence of the attenuation parameters on the individual receiver locations. Then the model becomes

$$\begin{aligned} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_M \end{bmatrix} &= \begin{bmatrix} e^{j\omega\tau_{11}} & e^{j\omega\tau_{12}} \\ e^{j\omega\tau_{21}} & e^{j\omega\tau_{22}} \\ \vdots & \vdots \\ e^{j\omega\tau_{M1}} & e^{j\omega\tau_{M2}} \end{bmatrix} \begin{bmatrix} a_1 s(\omega) \\ a_2 s(\omega) \end{bmatrix} + \begin{bmatrix} n_1(\omega) \\ \vdots \\ n_M(\omega) \end{bmatrix} \\ \mathbf{x} &= \mathbf{Ha} + \mathbf{n}, \end{aligned} \tag{3}$$

where the columns of $\mathbf{H}$ will be referenced as $\mathbf{h}_1$ and $\mathbf{h}_2$.

The Fisher information matrix for this model is given by, [5],

$$\mathbf{J}_\alpha = \frac{2}{\sigma^2} \int \mathrm{Re}\{\mathbf{D}_\alpha^t (\mathbf{I} - \mathbf{Q}) \mathbf{D}_\alpha\} d\omega, \tag{4}$$

where

$$\begin{aligned} \sigma^2 &= \text{standard deviation of the noise} && (5) \\ \mathbf{I} &= M \times M \text{ identity matrix} && (6) \\ \mathbf{Q} &= \mathbf{H}(\mathbf{H}^t\mathbf{H})^{-1}\mathbf{H}^t && (7) \\ \mathbf{D}_\alpha &= \left[\tfrac{\partial \mathbf{h}_1}{\partial \alpha} \quad \tfrac{\partial \mathbf{h}_2}{\partial \alpha}\right] \mathbf{a}. && (8) \end{aligned}$$

The Cramér-Rao bounds for the estimate of the range is given by the inverse of the Fisher information matrix, $\mathbf{J}_\alpha^{-1}$.

3. CONDITIONS WHERE MODEL IS UNRESOLVABLE

In most circumstances, the bounds in the previous section indicate an acceptable amount of error is possible providing that a suitable estimation technique is used. However, realistic conditions can occur that yield unacceptably high bounds for the given model. These conditions correspond to the situation where the vectors $\mathbf{h}_1$ and $\mathbf{h}_2$ become *nearly* collinear. We wish to emphasize that we are not focusing on the case where $\mathbf{H}$ is of rank 1, but rather the case where the vectors $\mathbf{h}_1$ and $\mathbf{h}_2$ are clearly linearly independent, yet only slightly non-collinear. In the landing problem, this might correspond to the case where the reflection results in only a very slight delay relative to the delay of the direct signal.

An example illustrates how these conditions can occur. Suppose that an airplane is located α m from the center of an antenna array on a carrier as shown in figure 1. The array consists of 4 receivers spaced 5 m apart located along the vertical. The signal from the airplane also bounces off of a reflector located 75 m from the center of the array at an angle of 45 degrees from vertical. The chosen signal-to-noise ratio (SNR) at the receivers is 20 dB for the direct component and 18 dB for the reflected component. The source operates at 10 GHz. Figure 2 depicts the error bounds for ranges between 200 m and 2500 m. As indicated in the figure, the error bounds suggest very large estimation errors as the range becomes larger. In general, this unresolvability will occur when the associated delays τ_{mk} of the two signals are nearly the same and the attenuation factors a_k are nearly the same as well.

4. IMPROVEMENTS IN THE ERROR PERFORMANCE

Since the Cramér-Rao bounds indicate the lower limit of the performance of any unbiased estimator, it is clear that we will not be able to find an estimator based on this model which generates accurate estimates when $\mathbf{h}_1$ and $\mathbf{h}_2$ become nearly collinear. A reformulation of the model into average and difference vectors suggests how to circumvent this problem. Using the average vector $\frac{1}{2}(\mathbf{h}_1 + \mathbf{h}_2)$ and the difference vector $\frac{1}{2}(\mathbf{h}_1 - \mathbf{h}_2)$, we can rewrite the model of (3) as

$$\begin{aligned} \mathbf{x} &= \left[\tfrac{1}{2}(\mathbf{h}_1 + \mathbf{h}_2) \quad \tfrac{1}{2}(\mathbf{h}_1 - \mathbf{h}_2)\right] \begin{bmatrix} (a_1 + a_2)s \\ (a_1 - a_2)s \end{bmatrix} + \mathbf{n} \\ \mathbf{x} &= \left[\mathbf{h}_+ \quad \mathbf{h}_-\right] \begin{bmatrix} a_+ s \\ a_- s \end{bmatrix} + \mathbf{n}. \end{aligned} \tag{9}$$

Since poor resolution occurs when $\mathbf{h}_1$ and $\mathbf{h}_2$ are nearly collinear and when the attenuation factors a_k are nearly the same, that corresponds to the case where $\mathbf{h}_-$ is nearly the zero vector and a_- is nearly 0. We will use this information to formulate an improvement in performance.

Rewrite the vectors $\mathbf{h}_+$ and $\mathbf{h}_-$ as

$$\mathbf{h}_+ = \begin{bmatrix} \cos(\frac{1}{2}\omega(\tau_{11}-\tau_{12}))e^{\frac{1}{2}j\omega(\tau_{11}+\tau_{12})} \\ \vdots \\ \cos(\frac{1}{2}\omega(\tau_{M1}-\tau_{M2}))e^{\frac{1}{2}j\omega(\tau_{M1}+\tau_{M2})} \end{bmatrix} \quad (10)$$

$$\mathbf{h}_- = \begin{bmatrix} j\sin(\frac{1}{2}\omega(\tau_{11}-\tau_{12}))e^{\frac{1}{2}j\omega(\tau_{11}+\tau_{12})} \\ \vdots \\ j\sin(\frac{1}{2}\omega(\tau_{M1}-\tau_{M2}))e^{\frac{1}{2}j\omega(\tau_{M1}+\tau_{M2})} \end{bmatrix} \quad (11)$$

The derivative of $\mathbf{h}_+$, $\mathbf{d}_+$, can be written as

$$\mathbf{d}_+ = \frac{1}{2}j\omega \begin{bmatrix} (\frac{\partial\tau_{11}}{\partial\alpha}+\frac{\partial\tau_{12}}{\partial\alpha})\left[\cos(\frac{1}{2}\omega(\tau_{11}-\tau_{12})) + j\frac{\frac{\partial\tau_{11}}{\partial\alpha}-\frac{\partial\tau_{12}}{\partial\alpha}}{\frac{\partial\tau_{11}}{\partial\alpha}-\frac{\partial\tau_{12}}{\partial\alpha}}\sin(\frac{1}{2}\omega(\tau_{11}-\tau_{12})\right]e^{\frac{1}{2}j\omega(\tau_{11}+\tau_{12})} \\ \vdots \\ (\frac{\partial\tau_{M1}}{\partial\alpha}+\frac{\partial\tau_{M2}}{\partial\alpha})\left[\cos(\frac{1}{2}\omega(\tau_{M1}-\tau_{M2})) + j\frac{\frac{\partial\tau_{M1}}{\partial\alpha}-\frac{\partial\tau_{M2}}{\partial\alpha}}{\frac{\partial\tau_{M1}}{\partial\alpha}+\frac{\partial\tau_{M2}}{\partial\alpha}}\sin(\frac{1}{2}\omega(\tau_{M1}-\tau_{M2})\right]e^{\frac{1}{2}j\omega(\tau_{M1}+\tau_{M2})} \end{bmatrix}. \quad (12)$$

Since poor resolution occurs when the delays of the two received signals are nearly equal, it is safe to assume that $\frac{1}{2}\omega(\tau_{m1}-\tau_{m2})$ is small for all m. Thus,

$$\sin(\frac{1}{2}\omega(\tau_{m1}-\tau_{m2})) \approx \frac{1}{2}\omega(\tau_{m1}-\tau_{m2}) \quad (13)$$

$$\cos(\frac{1}{2}\omega(\tau_{m1}-\tau_{m2})) \approx 1. \quad (14)$$

Furthermore, assume that $\frac{\frac{\partial\tau_{M1}}{\partial\alpha}-\frac{\partial\tau_{m2}}{\partial\alpha}}{\frac{\partial\tau_{m1}}{\partial\alpha}+\frac{\partial\tau_{m2}}{\partial\alpha}}$ is also small for all m. This can be shown to hold for the case of poor resolution. Then, substituting into (12) and (11) respectively,

$$\mathbf{d}_+ = \frac{1}{2}j\omega \begin{bmatrix} (\frac{\partial\tau_{11}}{\partial\alpha}+\frac{\partial\tau_{12}}{\partial\alpha})e^{\frac{1}{2}j\omega(\tau_{11}+\tau_{12})} \\ \vdots \\ (\frac{\partial\tau_{M1}}{\partial\alpha}+\frac{\partial\tau_{M2}}{\partial\alpha})e^{\frac{1}{2}j\omega(\tau_{M1}+\tau_{M2})} \end{bmatrix} \quad (15)$$

$$\mathbf{h}_- = \frac{1}{2}j\omega \begin{bmatrix} (\tau_{11}-\tau_{12})e^{\frac{1}{2}j\omega(\tau_{11}+\tau_{12})} \\ \vdots \\ (\tau_{M1}-\tau_{M2})e^{\frac{1}{2}j\omega(\tau_{M1}+\tau_{M2})} \end{bmatrix} \quad (16)$$

We also require that

$$\frac{\tau_{m1}-\tau_{m2}}{\frac{\partial\tau_{11}}{\partial\alpha}+\frac{\partial\tau_{12}}{\partial\alpha}} = \gamma \qquad \forall m, \quad (17)$$

where γ is a constant. This is again a fair assumption to make when the signal resolution is poor. Then,

$$\mathbf{h}_- = \gamma\mathbf{d}_+. \quad (18)$$

That is, $\mathbf{h}_-$ is collinear with $\mathbf{d}_+$.

Now, suppose that instead of using a model involving the two vectors $\mathbf{h}_+$ and $\mathbf{h}_-$, we base our estimation procedure on the model determined by the single vector $\mathbf{h}_+$. This will almost always result in a biased estimation technique, but since poor resolution occurs when $\mathbf{h}_+$ predominates, it is a reasonable model to consider. For a small change in α, $\triangle\alpha(\omega)$, the approximation

$$\mathbf{h}_+(\alpha+\triangle\alpha) = \mathbf{h}_+(\alpha) + \triangle\alpha\mathbf{d}_+(\alpha) \quad (19)$$

is valid. Define $\mathbf{y}_\triangle$ to be

$$\mathbf{y}_\triangle = \mathbf{h}_+(\alpha)a_+ + \triangle\alpha\mathbf{d}_+(\alpha)a_+. \quad (20)$$

The actual signal $\mathbf{y}$, consisting of components from $\mathbf{h}_+$ and $\mathbf{h}_-$, is given by

$$\mathbf{y} = \mathbf{h}_+(\alpha)a_+ + \mathbf{h}_-(\alpha)a_- \quad (21)$$

At any particular frequency ω, let us assume that we can find a shift $\triangle\alpha(\omega)$ such that

$$\mathbf{h}_+(\alpha+\triangle\alpha)a_+ = \mathbf{h}_+(\alpha)a_+ + \mathbf{h}_-(\alpha)a_-. \quad (22)$$

Equating (22) and (19), we see that, for a particular frequency ω, we can find an appropriate shift $\triangle\alpha(\omega)$ such that the single vector model and the two vector model are equivalent. This shift represents the bias due to estimation of the two component signal by the single vector model for one frequency. For the entire frequency band, we would like to find the best shift $\triangle\alpha(\omega)_b$ over all frequencies. This is accomplished by the least squares minimization

$$\triangle\alpha(\omega)_b = \text{argmin} \int |\mathbf{y}-\mathbf{y}_\triangle|^2 d\omega \quad (23)$$

The solution to this minimization is given by

$$\triangle\alpha(\omega)_b = \frac{\text{Re}\{\int(\mathbf{h}_-a_-)^t - \mathbf{d}_+a_+d\omega\}}{\int(\mathbf{d}_+a_+)^t\mathbf{d}_+a_+d\omega}. \quad (24)$$

This is the bias due to estimation of the two component signal by the single vector model for the entire frequency band. We can now characterize the total error using the single wave model as the sum of the error due to noise plus the error due to bias.

5. RESULTS AND CONCLUSION

In figure 3, the results using the same conditions as in figure 2 are plotted for the single wave model. As seen in the figure, the single signal model improves the error performance over the values of the range for this example, especially in areas where the two signal model fails miserably.

In general, the assumptions for the bias estimate are valid over only a portion of the total set of possible operating scenarios. That is, it is more appropriate to use the two signal model than the one signal model under most conditions. However, it is important to realize that there are realistic circumstances where a one signal model will significantly outperform the two signal model. While the models examined were simplfied in terms of their order, the same trend will be observed with higher order models. There will be biased, lower order models which yield lower errors. A similar type of analysis to that in the previous section should yield a bias estimate for a lower order model in such situations. An overall processing technique should incorporate both types of models. Using total error, one can determine the relative energy levels (projection onto the lower order subspace) where a lower order model is more suited. By measuring for the energy levels, the processing can incorporate the minimum error-producing model over the entire set of operating conditions.

6. ACKNOWLEDGEMENT

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7. REFERENCES

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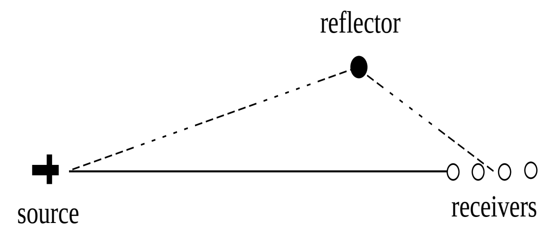

Figure 1: A diagram of the source, reflector, and receivers.

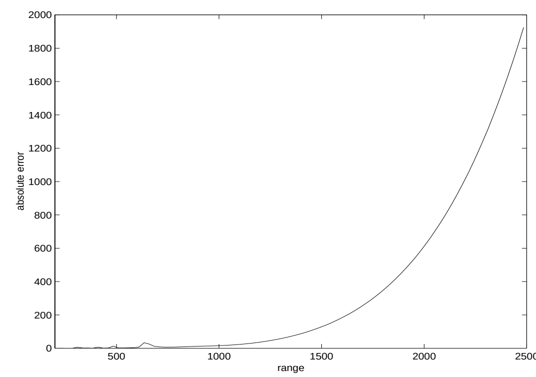

Figure 2: A plot of the error bounds for the given 2 signal model.

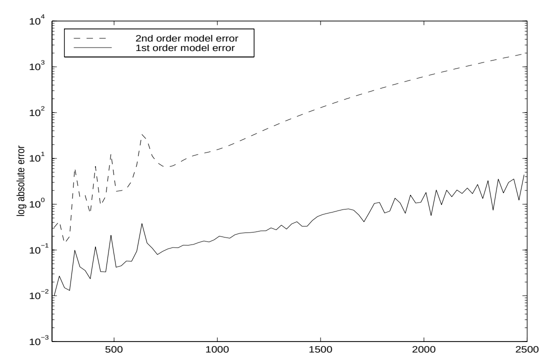

Figure 3: A comparison of the errors for the 2 signal model and the proposed model, plotted in log scale.