TIME-FREQUENCY DERIVATION OF PERIODIC WIDE BAND PROBING SIGNALS

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ABSTRACT

An analysis of a discrete time-frequency distribution yields a new periodic wide band probing signal for use in unknown system identification. The derivation is based on mathematical properties of the discrete Wigner distribution. Like the continuous distribution, the discrete version also satisfies the covariance property, meaning transformations in the time-frequency plane are equivalent to transformations in the time domain. By utilizing this property, the linearly swept frequency measurement is extended to discrete, periodic signals. The resulting probing signal possesses favorable characteristics such as a short illumination time requirement and good resistance to noise. The performance of the proposed probing method is compared with m-sequence methods and chirp signal methods.

1. INTRODUCTION

A common way to analyze an unknown linear system is to apply a known input signal and use processing techniques on the output signal to obtain information about the system, such as its frequency response. This probing signal method has found use in a variety of applications, including underwater acoustics, [1], room acoustics, [2], and digital mobile radio [3]. While the method is straightforward, its implementation depends on the type of probing signal chosen as an input. There has been much discussion in the literature about what type of signal to use, although there have not been many direct comparisons of the performance of different signals. With the advent of implemented digital systems, it is especially important to consider the performance of discrete-time signals. Furthermore, since we are typically interested in N samples of the frequency response, it is desirable to use N-periodic signals for probing systems.

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Two popular choices of probing signals are maximum-length sequences (or m-sequences, comprised of 1s and -1s in a specific order) and chirps (complex linear frequency-modulated signals of the form e^{jct^2} for some constant c). Each of these can be used to produce a wide-band frequency response estimate from a single measurement. An excellent discussion of m-sequences for use in probing signal applications can be found in [4, 5]. An m-sequence is suited for this purpose because its circular autocorrelation function is impulse-like if the length of the sequence is sufficiently long, [6]. A frequency response estimate is obtained by applying an m-sequence to a system, and then computing the Fourier transform of the circular autocorrelation of the output.

A practical implementation of chirps for probing systems is discussed in [7, 8]. In these references, an estimate is obtained by applying a sampled chirp (a discretized, continuous chirp signal) to a system, and then dividing the Fourier transform of the output by the Fourier transform of the input chirp. In the continuous domain, another implementation involving chirps has been presented in the context of the linearly swept frequency measurement, [9]. This measurement is computed by applying an analog chirp signal to a system, multiplying the output by another chirp, and then convolving that output with another chirp. However, the computational demands of this method when discretized are greater than those for m-sequences or the method given by [7, 8].

Beyond practical considerations, there is a theoretically useful connection between the linearly swept frequency measurement and time-frequency analysis. In [9, 10], Poletti illustrated how the (continuous) Wigner distribution could be used to analyze the steps involved in this measurement. By relating shears in the time-frequency plane with chirps in the time-domain, Poletti was able to suggest an improvement in the existing technique. The key to Poletti's analysis was that the continuous Wigner distribution satisfies the covariance property, which directly relates transformations in time-frequency to transformations in the time-domain.

Recently, a formulation of the discrete Wigner distribution was given which also satisfies the covariance property, [11]. Using this formulation, we derive a discrete-time, discrete-frequency linearly swept frequency measurement. The resulting probing signal method eliminates the computational disadvantages of the discretized continuous measurement of [9, 10]. We will then compare the performance of this new periodic, wide band probing signal with m-sequences and the chirps suggested by [7, 8].

2. PROPERTIES OF A GOOD PROBING SIGNAL

If there were no peak power limitations and no noise, the simplest way to obtain a frequency response estimate would be to send an impulse through the system and compute the Fourier transform of the output. In applications, however, these practical limitations must be considered. The suitability of a probing signal with respect to peak power restrictions and noise can be quantified by computing its crest factor (CF), which is defined for a signal s to be, [12],

$$CF(s) = \frac{||s||_{\inf}}{||s||_2} \geq 1. \tag{1}$$

Intuitively, a signal most impervious to noise for a given peak power limitation will have maximum amplitude at every sample, which would correspond to a crest factor of 1. A length-N impulse signal has a crest factor of $\sqrt{N}$. On the other hand, both m-sequences and chirp signals have a crest factor of 1, which explains why both are widely utilized.

In addition to crest factor, there are other practical considerations. One is the duration of the input signal. A shorter signal will require less power consumption and reduce the so-called illumination time of the probe. Another consideration is the amount of computation required for processing of the output signal. Also, for the case of a real (not complex), implemented system, a chirp signal requires two measurements to obtain an estimate due to its imaginary component. We will take all of these aspects into account when comparing the performance of these signals.

3. COVARIANCE AND A DISCRETE TIME-FREQUENCY PROBING SIGNAL

Covariance is a property that relates unit-norm linear transformations of coordinates of the Wigner distribution to unitary transformations in the time-domain. Specifically, if a given unit-norm linear transformation is applied to the Wigner distribution of a signal, there is a corresponding unitary transformation that can be applied to the signal in the time-domain which yields the Wigner distribution of the transformed signal. Such unit-norm linear transformations of the time-frequency plane are called symplectic transformations. This property is stated mathematically as follows: applying a symplectic transformation, $\mathcal{A}$, to a Wigner distribution W_x is equivalent to first applying a unitary transformation which depends on $\mathcal{A}$, $\mathcal{U}(\mathcal{A})$, to x, and then computing the Wigner distribution i.e.

$$W_x(\mathcal{A}(t,f)) = W_{\mathcal{U}(\mathcal{A})x}(t,f). \tag{2}$$

In [9, 10], the frequency response of a system with impulse response $h(t)$ is

$$H(t) = e^{-\frac{j\pi}{4}} \left[\left[h(t) * e^{j\pi t^2}\right] e^{-j\pi t^2}\right] * e^{j\pi t^2},. \tag{3}$$

The algorithm implied by this equation has three steps. In the first step, the input chirp signal is convolved with the impulse response. Convolution of $h(t)$ with a chirp corresponds to a shear in the time-frequency plane. That is, if $s(t) = e^{j\pi t^2}$, then

$$W_h(\mathcal{A}_s(t,f)) = W_{\mathcal{U}(\mathcal{A}_s)h}(t,f), \tag{4}$$

where

$$\mathcal{A}_s = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}. \tag{5}$$

The second step involves multiplication by the chirp $s^*(t)$. This corresponds to the transformation of time-frequency coordinates given by $\mathcal{A}_s^{-1}$. The third step is again convolution with $s(t)$. The overall sequence of shearing transformations involved in a swept frequency measurement is given by the composition

$$\mathcal{A}_s \cdot \mathcal{A}_s^{-1} \cdot \mathcal{A}_s \tag{6}$$

We now extend this concept to the discrete-time, discrete-frequency domain. The impulse response of the sytstem is given by $h(n)$, and the goal is to compute N frequency samples of this system. In the discrete domain, the correspondence between chirps in time and shears in time-frequency occurs when N is odd, [13], so we will restrict our discussion to this case. In the discrete time-frequency plane, we want to apply the same sequence of shearing operations as given in (6). The corresponding time domain operation is *not* a sampled version of the continuous chirp signal. Instead, the "chirp" corresponding to $\mathcal{A}_s$ for discrete-time, discrete-frequency is the N-periodic signal given by, [14],

$$s_d(n) = e^{\frac{j2\pi(2^{-1}n^2)_N}{N}}, \tag{7}$$

where 2^{-1} is the group-theoretic inverse of 2 in the group of integers modulo N, and the notation $(a)_N$ means $a \mod N$. Also, the time-domain operation is not regular convolution, but *circular* convolution, [14]. Thus, the sequence of shearing transformations from (6) corresponds to the discrete swept-frequency measurement

$$H(n) = \lambda\,[[h(n) \circledast s_d(n)]\, s_d^*(n)] \circledast s_d(n), \tag{8}$$

where $\lambda = \frac{1}{\sqrt{N}} e^{\frac{j\pi[(N)_8 - 1]}{4}}$. The algorithm implied by (8) requires further explanation because of the initial circular convolution. Since the system $h(n)$ can only be probed via a standard filtering operation (regular convolution), we need to modify the first step of the discrete swept-frequency measurement. Since circular convolution is equivalent to regular convolution followed by time-aliasing, [15], the output of the system needs to be time-aliased before subsequent processing. The resulting processing scheme is presented in the block diagram in figure 1.

4. COMPARISON OF METHODS

In this section, we compare the performance of m-sequences and sampled chirps with that of the group-theoretic chirp (the signal in (7)). For each method, we generate $N = 16$ equally spaced samples of the frequency response (corresponding to a length-31 probing signal) for the length-155 linear, time-invariant low pass filter depicted in figure 2. We add white Gaussian noise to the output signal prior to the post-processing steps to simulate noisy conditions. Since the system $h(n)$ is real, we will penalize the complex probing signal methods (which effectively require two measurements) by comparing a single complex probing signal estimate with the average of two m-sequence estimates. For each method, we compute the absolute error of the average of 100 estimates (or 200 for the m-sequence estimates).

Figure 2 contains the $N = 16$ sample estimates of the frequency response $H(k)$ computed via the proposed method. A scatter plot of the absolute error for estimates obtained by the three probing signal methods is shown in figure 3. The results for the proposed method and the m-sequence method are both good (and roughly equivalent). In this example, the sampled chirp method does not yield satisfactory results because it does not compute equally spaced samples of the frequency response when the desired number of samples is less than the length of the filter. We do note that when the number of samples desired is greater than the length of the filter, all three probing signal methods produce very similar error performance.

We now examine some other considerations. First, as given by [8], the sampled chirp probing method requires an illumination time equal to 2 periods of the sampled chirp (i.e. $2N$). The group-theoretic chirp probing method and the m-sequence method require only 1 period. We make the important observation, however, that the sampled chirp probing method can be modified so that only 1 period is required for an estimate by time-aliasing the output of a 1 period input sampled chirp before further processing. Computationally, the post-processing demands of each method are roughly similar. For the m-sequence method, post-processing entails the computation of the length-N circular auto-correlation function followed by a length-N discrete Fourier transform. The sampled chirp method (with reduced illumination time) requires a time-alias by N step, a length-N discrete Fourier transform, and then a length-N division. The group-theoretic chirp method involves a time-alias by N step, a length-N multiplication, and a length-N circular convolution. Finally, the m-sequence method possesses a further restriction that the number of samples obtained satisfy $N = 2^M - 1$ for some positive integer M (which is why the example chosen satisfied this requirement).

5. CONCLUSION

The results of the preceding section suggest a few conclusions regarding choice of a discrete, periodic wide-band probing signal. When only a small number of samples are required (i.e. a short illumination time), either the proposed method or the m-sequence method is suitable. If the number of samples desired cannot be expressed as $N = 2^M - 1$, the proposed probing signal method should be used. When a large number of samples of a real input/output system are desired, any of the methods will yield good performance, although some of our work indicates that m-sequences do slightly better. We also mention that if the system under study happens to be complex input/output, then the complex probing signal methods would be preferred.

While the method proposed here performs satisfactorily, a more significant result is that we have described a practical application of mathematical properties relevant to time-frequency analysis. The group-theoretic chirp signal is a direct result of study of the covariance property for the discrete Wigner distribution. Many articles in the literature have focused on the importance of various time-frequency properties.

The authors hope we have demonstrated that covariance is such a property due to its usefulness in an actual application.

6. REFERENCES

[1] G. E. J. Bold, M. D. Johns, and T. G. Birdsall, "Signals for optimum bandlimited system interrogation with application to underwater acoustics," *Journal of the Acoustic Society of America*, vol. 81, pp. 991–999, April 1987.

[2] M. R. Schroeder, "Integrated-impulse method measuring sound decay without using impulses," *Journal of the Acoustical Society of America*, vol. 66, pp. 497–500, August 1979.

[3] H. W. Schüssler and U. Martin, "Devices for measuring and simulating the mobile radio channel," in *Proc. IEEE DSP Workshop*, pp. 45–48, 1994.

[4] D. D. Rife and J. Vanderkooy, "Transfer-function measurement with maximum-length sequences," *Journal of the Audio Engineering Society*, vol. 37, pp. 419–444, June 1989.

[5] J. Vanderkooy, "Aspects of mls measuring systems," *Journal of the Audio Engineering Society*, vol. 42, pp. 219–231, April 1994.

[6] S. W. Golomb, *Shift Register Sequences*. Aegean Park Press, 1982.

[7] P. Vary, "Fast digital frequency response measurement with multifrequency signals," *Arch. el. Übertr.*, vol. 34, no. 5, pp. 190–195, 1980.

[8] H. W. Schüssler and F. Heinle, "Measuring the properties of implemented digital systems," *Frequenz*, vol. 48, pp. 3–7, April 1994.

[9] M. A. Poletti, "Linearly swept frequency measurements, time-delay spectrometry, and the Wigner distribution," *Journal of the Audio Engineering Society*, vol. 36, pp. 457–468, June 1988.

[10] M. A. Poletti, "The application of linearly swept frequency measurements," *Journal of the Acoustic Society of America*, vol. 84, pp. 599–610, August 1988.

[11] M. S. Richman, T. W. Parks, and R. G. Shenoy, "Discrete-time, discrete-frequency time-frequency representations," in *Proc. Int. Conf. Acoust., Speech, Sig. Proc.*, May 1995.

[12] S. Boyd, "Multitone signals with low crest factor," *IEEE Transactions on Circuits and Systems*, vol. CAS-33, pp. 1018–1022, October 1986.

[13] M. S. Richman, T. W. Parks, and R. G. Shenoy, "Discrete-time, discrete-frequency, time-frequency analysis." Submitted to IEEE Trans. Signal Processing, preprint available at http://cam.cornell.edu/richman/discwig.ps.Z.

[14] M. S. Richman, T. W. Parks, and R. G. Shenoy, "Features of a discrete wigner distribution," in *Proc. IEEE DSP Workshop*, September 1996.

[15] A. V. Oppenheim and R. W. Schafer, *Discrete-time signal processing*. Prentice Hall, 1989.

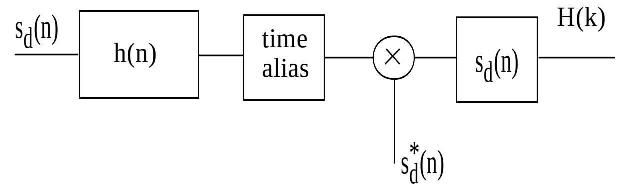

Figure 1: A block diagram of a discrete linearly swept frequency measurement.

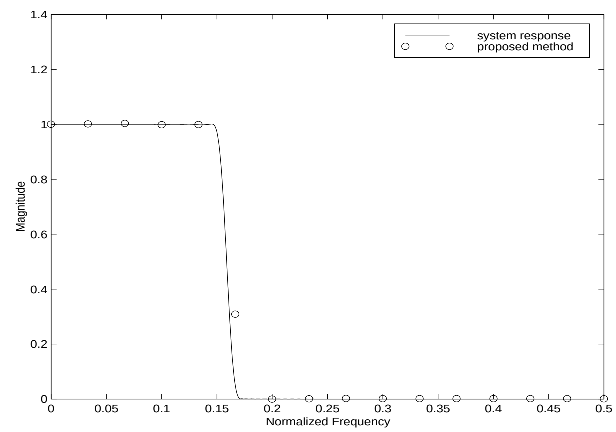

Figure 2: The system $H(k)$ to be estimated along with $N = 16$ sample estimates generated by the proposed method.

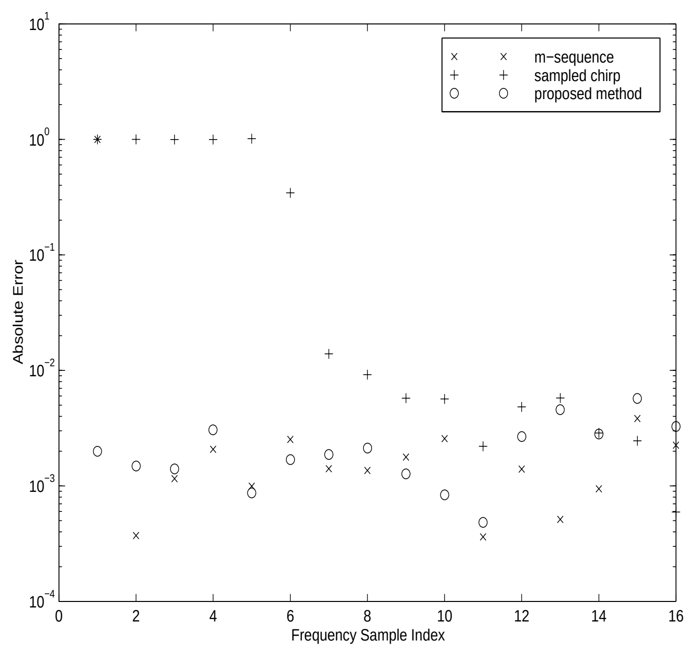

Figure 3: A comparison of the absolute error of the three methods for $N = 16$ equally spaced DFT samples.