OPTIMAL PULSE SHAPE FOR ESTIMATING POSITIONS OF SUPERIMPOSED PULSES

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ABSTRACT

In this paper, we determine the optimal pulse shape for estimating positions of superimposed pulses by deriving the Cramer-Rao lower bound (CRLB) on the average estimation error variance and optimizing it with respect to pulse shape. Our results show that a significant improvement in estimation error variance can be achieved relative to Gaussian and rectangular pulse shapes.

1. INTRODUCTION

We consider the problem of estimating pulse positions within a train of superimposed pulses. We formulate the measurement of each individual pulse position as a multi-dimensional least-squares estimation problem, and determine the pulse shape that minimizes the Cramer-Rao lower bound (CRLB) for the average estimation error variance. Our result can be used in applications such as magnetic resonance (MR) cardiac tagging [1] and pulse-position modulation (PPM) [2] where a signal is encoded by the pulse displacement from a specified position reference. Atalar and McVeigh [3] addressed a similar problem in the context of MR tagging where the position estimation of a single pulse is considered. In particular, the optimal pulse thickness was determined for several typical pulse shapes. Here we extend this problem to multiple superimposed pulses. Specifically, we derive an analytical expression for the CRLB of the average estimation error. We then simplify this expression for the case of equally spaced pulses. The optimal pulse shape is found by optimizing this bound over the class of bandlimited pulses with an upper bound on the pulse energy.

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This paper is organized as follows. In Section 2, we derive the CR bound matrix. The analytical derivation for the CRLB is then given in Section 3. In Section 4, we optimize the CRLB numerically and present some simulation results for both the case of equally and non-equally spaced pulses. Conclusions and directions for future work are presented in Section 5.

2. PRELIMINARIES

2.1. Notation

We denote a 1-D individual pulse function by $s(x)$, where x is the independent variable which could be time or spatial position. We assume that the pulse has passed through a physical system with finite bandwidth (e.g., a communication channel). For the case of a pulse sequence, the superimposed pulse signal is given by

$$y(x) = \sum_{i=1}^{M} s(x - \mu_i) \tag{1}$$

where $\mu_i, i = 1, \ldots, M$, denotes the position of each individual pulse center.

In order to simplify the subsequent analytical development, we introduce vector notation as follows. Given the pulse positions $\boldsymbol{\mu} = [\mu_1, \mu_2, \ldots, \mu_M]$ and the sampling locations $\mathbf{x} = [x_1, x_2, \ldots, x_N]$, the uncorrupted samples will be represented by a vector $\mathbf{y}(\mathbf{x}, \boldsymbol{\mu})$ whose elements are given by

$$y_n = y(x_n, \boldsymbol{\mu}) = \sum_{i=1}^{M} s(x_n - \mu_i). \tag{2}$$

Similarly, the observed data samples are collected into a vector $\tilde{\mathbf{y}}(\mathbf{x}, \boldsymbol{\mu})$. Our observation model is given by $\tilde{\mathbf{y}}(\mathbf{x}, \boldsymbol{\mu}) = \mathbf{y}(\mathbf{x}, \boldsymbol{\mu}) + \mathbf{n}(\mathbf{x})$, where $\mathbf{n}(x)$ denotes a Gaussian noise vector with zero mean and known correlation matrix Λ which is uncorrelated with the uncorrupted signal $\mathbf{y}(\mathbf{x}, \boldsymbol{\mu})$.

2.2. The CR Bound Matrix

In the following we derive the CR bound matrix for the problem of estimating pulse positions within a superimposed pulse sequence given in (1). Given the pulse positions $\boldsymbol{\mu}$ and the sampling locations $\mathbf{x}$, the Fisher information matrix Γ is given as follows:

$$\Gamma = E\left\{\left[\frac{\partial \ln f(\tilde{\mathbf{y}}|\mathbf{x}, \boldsymbol{\mu})}{\partial \boldsymbol{\mu}}\right]\left[\frac{\partial \ln f(\tilde{\mathbf{y}}|\mathbf{x}, \boldsymbol{\mu})}{\partial \boldsymbol{\mu}}\right]^T\right\} \quad (3)$$

where

$$f(\tilde{\mathbf{y}}|\mathbf{x}, \boldsymbol{\mu}) = \frac{1}{(2\pi)^{N/2}|\det \Lambda|^{1/2}} \exp[-\frac{1}{2}(\tilde{\mathbf{y}}-\mathbf{y})\Lambda^{-1}(\tilde{\mathbf{y}}-\mathbf{y})^T] \quad (4)$$

and $E\{\cdot\}$ denotes the expectation operator. The CR bound matrix for the estimation error covariance is given by Γ^{-1}.

After simple mathematical manipulations, the information matrix can be reduced into the following convenient form:

$$\Gamma = \left[\frac{\partial \mathbf{y}}{\partial \boldsymbol{\mu}}\right]^T \Lambda^{-1} \left[\frac{\partial \mathbf{y}}{\partial \boldsymbol{\mu}}\right] \quad (5)$$

where $\left[\frac{\partial \mathbf{y}}{\partial \boldsymbol{\mu}}\right]$ is an $N \times M$ matrix, and Λ^{-1} is an $N \times N$ matrix. Taking into account (2), the ijth element of Γ is given by

$$\Gamma_{ij} = \left[\frac{\partial}{\partial \mu_i} s(x_n - \mu_i)\right]^T \Lambda^{-1} \left[\frac{\partial}{\partial \mu_j} s(x_n - \mu_j)\right] \quad (6)$$

where $n = 1, \ldots, N$. If we diagonalize and normalize the noise covariance matrix Λ (i.e., whitening the observed data by an appropriate transformation), Γ_{ij} is further reduced to

$$\Gamma_{ij} = \frac{1}{\sigma_n^2}\left[\frac{\partial}{\partial \mu_i} s(x_n - \mu_i)\right]^T \left[\frac{\partial}{\partial \mu_j} s(x_n - \mu_j)\right] \quad (7)$$

where σ_n^2 is the noise variance, and $s(x)$ now denotes the transformed version of the original individual pulse. We will use this notation for the transformed pulse shape hereafter.

3. AVERAGE ESTIMATION ERROR VARIANCE

3.1. Analytical Development

In this section, we derive an analytical expression for the CRLB on the average estimation error. We assume an infinite number of pulses ($M \to \infty$) and data samples ($N \to \infty$). We also assume that the pulse positions $\{\mu_i, i = 1, \ldots, M\}$ are equally spaced with spacing D. Furthermore, we are only concerned with the bandlimited pulses as mentioned above. Let Δx denote the critical sampling associated with the bandlimited pulse by which we sample the signal. The CRLB on the average estimation error variance is defined as follows:

$$\begin{aligned} \bar{\sigma}_{CR}^2 &= \lim_{M\to\infty} \frac{1}{M} tr[\Gamma^{-1}] \\ &= \lim_{M\to\infty} \frac{1}{M} \sum_{i=1}^{M} \lambda_i \\ &= \lim_{M\to\infty} \frac{1}{M} \sum_{i=1}^{M} \frac{1}{\rho_i} \end{aligned} \quad (8)$$

where ρ_i and $\lambda_i, i = 1, \ldots, M$, are the eigenvalues of the information matrix Γ and its inverse, respectively.

To derive an expression for $tr[\Gamma^{-1}]$, we first show that Γ is a Toeplitz matrix. Define the autocorrelation function $r(d)$ of the derivative pulse function $s'(x)$ as

$$r(d) = \int_{-\infty}^{\infty} s'(x)s'(x-d)dx \quad (9)$$

where $s'(x) = ds/dx$. Similarly, define the autocorrelation sequence $\Gamma[k]$ of the same process given its samples as

$$\Gamma[k] = \sum_{n=-\infty}^{\infty} s'(x_n)s'(x_n - kD). \quad (10)$$

From the sampling theorem, we have the relationship $\Gamma[k] = \frac{1}{\Delta x} r(kD)$ for any bandlimited pulse function. We can then express (7) in terms of (9) and (10) as follows: $\Gamma_{ij} = \frac{1}{\sigma_n^2}\Gamma[i-j] = \frac{1}{\sigma_n^2} r((i-j)D)$. As a result, we have shown that for the limiting case of an infinite number of equally spaced pulses and data samples, Γ is a symmetric Toeplitz matrix. This is true assuming that there is no aliasing.

By applying the Eigenvalue Distribution Theorem [4] to (8), the CRLB is given by

$$\bar{\sigma}_{CR}^2 = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{\sigma_n^2}{\Gamma(\omega)} d\omega \quad (11)$$

where ω denotes the discrete-space frequency variable, and $\Gamma(\omega)$ represents the Fourier transform of the autocorrelation sequence $\Gamma[k]$ defined in (10).

We now relate $\Gamma(\omega)$ to the Fourier transform of the continuous pulse shape $S(\Omega)$, where Ω is the continuous-space frequency variable. Define the autocorrelation sequence $\tilde{\Gamma}[l]$ computed using samples with critical spacing Δx as follows:

$$\tilde{\Gamma}[l] = \sum_{-\infty}^{\infty} s'(x_n)s'(x_n - l\Delta x) \quad (12)$$

Note that $, [k] = \tilde{,} [k\frac{D}{\Delta x}]$, i.e., $, [k]$ can be expressed as a downsampled version of $\tilde{,} [l]$. For convenience we assume that D is an integral multiple of Δx. Taking the Fourier transform on both sides of (12) we have

$$\tilde{,} (\omega) = \frac{1}{\Delta x} \sum_{n=-\infty}^{\infty} (\frac{\omega - 2\pi n}{\Delta x})^2 \left| S(\frac{\omega - 2\pi n}{\Delta x}) \right|^2 . \quad (13)$$

Note the effect of the derivative operator on the Fourier transform of $s'(x)$. Now taking into account the downsampling operation, $, (\omega)$ is given by

$$, (\omega) = \frac{1}{D} \sum_{n=-\infty}^{\infty} (\frac{\omega - 2\pi n}{D})^2 \left| S(\frac{\omega - 2\pi n}{D}) \right|^2 . \quad (14)$$

By substituting the previous result for $, (\omega)$ into (11), the final result for the CRLB on the average estimation error variance is given by the following:

$$\bar{\sigma}_{CR}^2 = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{D \Delta x \sigma_n^2}{\sum_{n=-\infty}^{\infty} (\frac{\omega - 2\pi n}{D})^2 \left| S(\frac{\omega - 2\pi n}{D}) \right|^2} d\omega \quad (15)$$

Note that the CRLB is well-defined if and only if the denominator of the integral in (15) does not vanish for any frequency between $-\pi$ and π.

Several important observations can be made from the analytical expression for the CRLB derived above. First, we should note that given a bandlimited pulse, the pulse spacing must be greater than 2 critical samples; otherwise the sum in the denominator of the integral in (15) will become zero at some frequencies.

In [3], Atalar and McVeigh had derived the CR bound for the problem of estimating a single pulse position. It can be shown that the optimal pulse function for this bound is a sinusoid with frequency equal to the system bandwidth. However, this pulse function does not work for the multiple pulse position problem because $S(\omega)$ would be a delta function, which makes the denominator of the integral in (15) become zero at certain frequencies. Intuitively the sinusoid fails in the multiple-pulse case because a sum of shifted sinusoids is also a sinusoid, and the individual pulses cannot be resolved.

Finally, one can show that the multiple-pulse CRLB in (15) reduces to the single pulse CR bound derived in [3], as the pulse spacing D approaches infinity.

3.2. Pulse Shape Optimization

We define the optimal pulse shape $s^*(x)$ as the $s(x)$ that minimizes the CRLB derived in (15) subject to the energy and bandwidth constraint for a given pulse spacing D. The energy constraint is needed to keep the CRLB from going to zero during optimization. The finite bandwidth constraint is reasonable in practice and is already assumed in deriving the CRLB in (15).

4. EXPERIMENTS

We demonstrate our method by numerically optimizing the CRLB for the case $D = 4$ critical samples. The optimization is done in the discrete frequency domain for 64 samples. The optimal pulse function and its DFT are given in Figures 1 and 2, respectively.

In order to verify our result, we generated a signal of 21 superimposed pulses spaced $D = 4$ samples apart, and convolved each pulse with a sinc function to simulate the finite bandwidth constraint. White Gaussian noise was added to the signal and the pulse positions were estimated using a non-linear least-squares fit. The estimation error averaged over 250 trials along with the CRLB is plotted in Figure 3. Also shown are the results of the same experiment using rectangular (1.5 samples wide) and Gaussian (1.2 samples wide) pulse shapes. The optimal thicknesses of the rectangular and Gaussian pulses were computed by minimizing (15) over pulse thickness. The amplitudes of the rectangular and Gaussian pulses were adjusted so that the pulses have an energy of 1. Our result shows that the optimal pulse function gives a better performance relative to rectangular and Gaussian pulses, particularly at low SNR. This is because it focuses the pulse energy in the available system bandwidth. More importantly, it concentrates the energy in those frequencies that are relevant to the estimation problem, which also results in increased bandwidth efficiency.

Finally, we performed computer experiments in order to investigate the performance when the assumption of even pulse spacing does not hold. This was done by displacing an equally spaced pulse sequence of 21 pulses ($D = 4$ critical samples) with a random signal uniformly distributed with known maximum displacement, and computing the average estimation error variance after 250 trials. The SNR was fixed at 25 dB. The result as a function of the maximum displacement of the random shift can be seen in Figure 4. Our result shows that the optimal pulse function still gives better results compared to rectangular and Gaussian pulses, even when the equal spacing assumption is violated.

5. CONCLUSIONS

In this paper we have derived the CRLB on the average estimation error variance in measuring positions of equally spaced pulses in a pulse sequence. We have shown that in order to keep the bound well-defined, the spacing must be larger than 2 critical samples. By optimizing the CRLB numerically we have been able to find the theoretical optimal pulse shape for a given pulse spacing. We note that other constraints such as a maximum amplitude or nonnegativity constraint could also be used in the optimization if desired. Simulation results have confirmed that the optimal pulse function gives a better performance relative to rectangular and Gaussian pulses, even for perturbations from equal spacing. One characteristic of our optimal pulse shape is that because it is well localized in frequency, it is not localized in space. In our future work we plan to add a spatial localization constraint in the optimization and develop other shift models that incorporate prior knowledge of the pulse displacements.

6. REFERENCES

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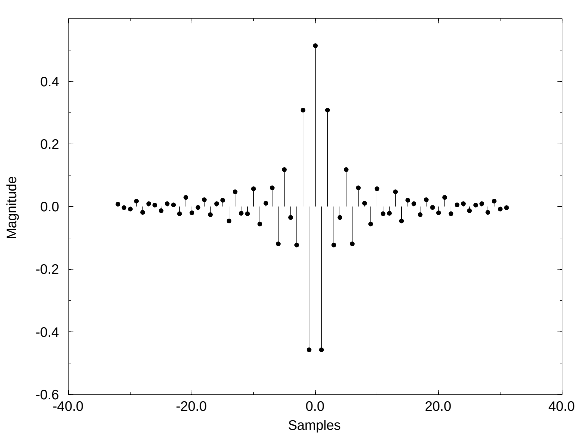

Figure 1: Optimal pulse function

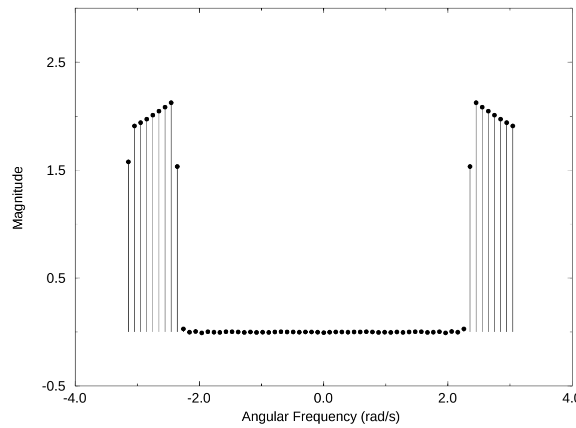

Figure 2: Fourier transform of optimal pulse function

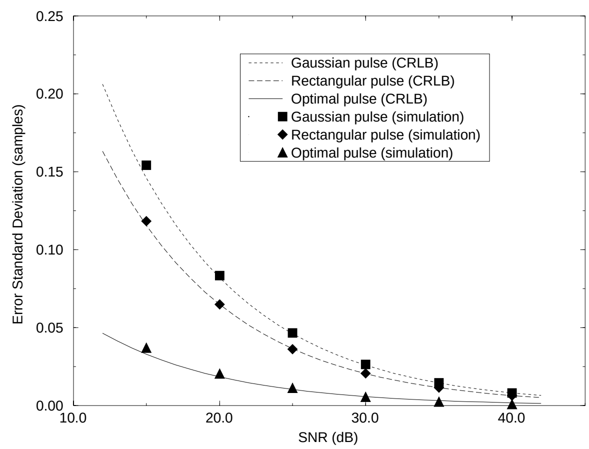

Figure 3: Average deviation of the estimation error vs SNR.

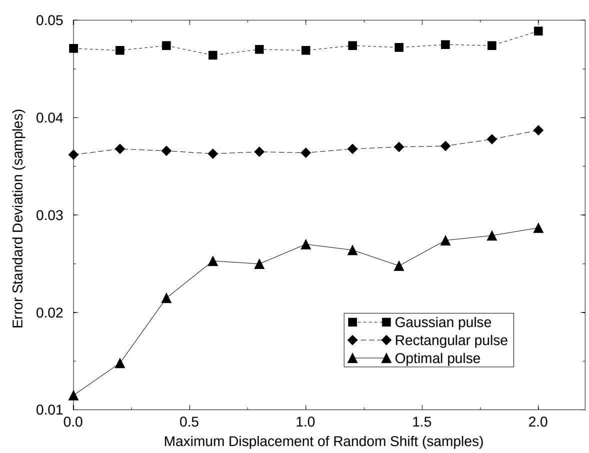

Figure 4: Average deviation of the estimation error vs maximum displacement .