GENERICALLY SUFFICIENT CONDITIONS FOR EXACT MULTICHANNEL BLIND IMAGE RESTORATION

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ABSTRACT

We have previously developed an algorithm and sufficient conditions for exact multichannel blind image restoration. In this paper, we use the resultant matrix theorem and techniques of algebraic geometry to prove that the sufficient conditions hold generically given three blurred versions of the same image and some restrictions on the size of the original image. Moreover, the extension to multichannel blind n-dimensional signal restoration is described.

1. INTRODUCTION

Recently, we extended a blind, one-dimensional symbol estimation algorithm [1] to two dimensions [2], i.e., multichannel blind image restoration. We have proved sufficient conditions to achieve exact restoration of blurred images in the noise-free case; i.e., the restored image is the same as the original image up to a scalar multiplier. A more advanced discussion of this algorithm in the noisy case and a correction to the sufficient conditions are provided in [3]. Here, we use the resultant matrix theorem and techniques of algebraic geometry to prove that the sufficient conditions hold generically given three blurred versions of the same images and some restrictions on the size of the original image.

While we were completing this work, we became aware of similar results obtained by Harikumar and Bresler [4]. Nevertheless, their arguments and algorithm are different from ours. Our approach can be easily extended to multichannel blind n-dimensional signal restoration.

The remainder of this paper is organized as followed. Section 2 describes a model for a multichannel imaging system (noise-free). In Section 3, we mention two necessary definitions and two useful theorems for our main theorems, which are proved in Section 4. The extension to multichannel blind n-dimensional signal restoration is in Section 5. We present simulation results in Section 6. Conclusions, with a description of future work on this topic, are made in Section 7.

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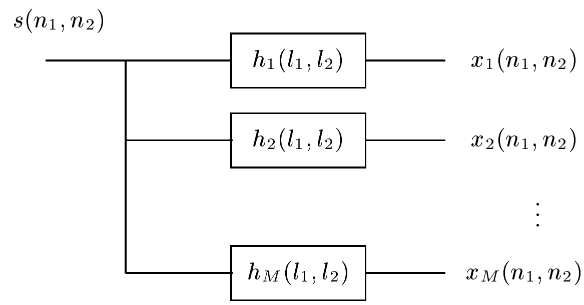

Figure 1: Single-input multiple-output image-blur model

2. PROBLEM STATEMENT

As shown in Figure 1, in the noise-free case the mth observed image is given by

$$x_m(n_1, n_2) = \sum_{l_1=0}^{L_1-1} \sum_{l_2=0}^{L_2-1} h_m(l_1, l_2) s(n_1 - l_1, n_2 - l_2) \ ,$$

where $s(n_1, n_2)$ is the original image and $x_m(n_1, n_2)$ is the output from the mth linear, space-invariant blur function $h_m(l_1, l_2)$. The extent of $s(n_1, n_2)$ is $N_1 \times N_2$. The maximum orders of all blur functions in each dimension are L_1 and L_2, respectively. There are M observed images. Therefore, $n_j = 0, 1, \ldots, N_j$; $l_j = 0, 1, \ldots, L_j$; $j = 1, 2$; and $m = 1, 2, \ldots, M$.

3. BACKGROUND DEFINITIONS AND THEOREMS

First, we make some definitions which are similar to those in many algebraic geometry texts. $\mathbb{C}$ denotes the complex numbers.

- Definition 1 ([5]): A subset $V \subseteq \mathbb{C}^n$ is an *algebraic set* if there is a collection $\{f_i\}$ of complex polynomials in n variables such that V is the set of common zeros of $\{f_i\}$. In other words,

$$V = \{\mathbf{x} = (x_1, x_2, \ldots, x_n) \mid \forall i : f_i(\mathbf{x}) = 0\} \ .$$

- Definition 2 ([6]): A subset $U \subseteq \mathbb{C}^n$ is *generic* if its complement is contained in an algebraic set whose dimension is less than n.

Remark: Let $U_1, U_2, \ldots, U_k$ be generic subsets of $\mathbb{C}^n$, where k is a finite number, and let the complement of U_i be contained in the algebraic set V_i. Define $U = \bigcap U_i$. Then the complement of U is contained in $V = \bigcup V_i$, whose dimension is less than n ([7], page 67). Therefore, U is generic.

Theorem 1 ([7], page 61): If V is an algebraic subset of $\mathbb{C}^n$, then $dim(V)$ (the dimension of V) is less than or equal to n. If $dim(V)$ is equal to n then $V = \mathbb{C}^n$.

Given two polynomials

$$a(z) = a_0 z^{p-1} + a_1 z^{p-2} + \cdots + a_{p-1},\ a_0 \neq 0$$

and

$$b(z) = b_0 z^{q-1} + b_1 z^{q-2} + \cdots + b_{q-1},\ q \leq p,$$

define a so-called *resultant matrix* or *Sylvester matrix*

$$R(\{a_i, b_j\}) = \underbrace{\begin{bmatrix} a_0 & a_1 & \cdots & a_{p-1} & 0 & \cdots & 0 \\ 0 & a_0 & a_1 & \cdots & a_{p-1} & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & a_0 & a_1 & \cdots & a_{p-1} \\ b_0 & b_1 & \cdots & b_{q-1} & 0 & \cdots & 0 \\ 0 & b_0 & b_1 & \cdots & b_{q-1} & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & b_0 & b_1 & \cdots & b_{q-1} \end{bmatrix}}_{p+q-2 \text{ columns}} .$$

Note that $R(\{a_i, b_j\})$ is a $(p+q-2) \times (p+q-2)$ square matrix.

Theorem 2 (*Sylvester's Resultant*) ([8]): The polynomials $a(z), b(z)$ share no common zero if and only if $det(R(\{a_i, b_j\}))$ (the determinant of $R(\{a_i, b_j\})$) is not equal to 0.

In other words, $det(R(a_i, b_j))$ is equal to 0 if and only if a_0 is equal to zero or $a(z), b(z)$ do share a common zero. In this paper, we call a_0 the leading element of the resultant matrix.

4. MAIN THEOREMS

Define that $S_{n_1} =$

$$\begin{bmatrix} s(n_1, 0) & s(n_1, 1) & \cdots & s(n_1, N_2 - r_2 - 1) \\ s(n_1, 1) & s(n_1, 2) & \cdots & s(n_1, N_2 - r_2) \\ \vdots & \vdots & \vdots & \vdots \\ s(n_1, r_2) & s(n_1, r_2 + 1) & \cdots & s(n_1, N_2 - 1) \end{bmatrix},$$

where $r_2 = L_2 + K_2 - 2$. In [2, 3], we prove the following

Theorem 3: Assume that

1. The polynomials

$$h_m(z_1, z_2) = \sum_{l_1=0}^{L_1-1} \sum_{l_2=0}^{L_2-1} h_m(L_1 - l_1 - 1, L_2 - l_2 - 1) z_1^{l_1} z_2^{l_2}$$

for $m = 1, 2, \ldots, M$ share no common zero;

2. The $h_m(0, 0)$ terms for $m = 1, 2, \ldots, M$ are not all zero;
3. The polynomials $\sum_{l_1=0}^{L_1-1} h_m(L_1 - l_1 - 1, 0) z_1^{l_1}$ for $m = 1, 2, \ldots, M$ share no common zero;
4. The polynomials $\sum_{l_2=0}^{L_2-1} h_m(0, L_2 - l_2 - 1) z_2^{l_2}$ for $m = 1, 2, \ldots, M$ share no common zero; and
5. $\mathbf{S}(r_1+1, r_2)$ and $\mathbf{S}(r_1, r_2+1)$ have full row rank, where $r_1 = L_1 + K_1 - 2$ and

$$\mathbf{S}(r_1, r_2) = \begin{bmatrix} S_0 & S_1 & \cdots & S_{N_1 - r_1 - 1} \\ S_1 & S_2 & \cdots & S_{N_1 - r_1} \\ \vdots & \vdots & \vdots & \vdots \\ S_{r_1} & S_{r_1+1} & \cdots & S_{N_1 - 1} \end{bmatrix}.$$

Then the original image s can be exactly restored, up to a scalar ambiguity, by choosing $K_1 > (L_1 - 1) r_2$ and $K_2 > L_2 - 1$.

Now, we prove that the conditions above hold generically if $M \geq 3$ and the size of s is large enough.

Lemma 1: Conditions 1–4 hold generically for $M \geq 3$.

Proof: Obviously, if conditions 1-4 can be satisfied when $M = 3$, then they also can be satisfied when $M > 3$. Therefore we assume $M = 3$.

First, we prove that condition 1 is satisfied generically. Let us rewrite $h_m(z_1, z_2)$ as

$$h_m(z_1, z_2) = \sum_{l_2=0}^{L_2-1} h_m^{l_2}(z_1) z_2^{l_2}, \tag{1}$$

i.e., as a one-variable (z_2) polynomial with coefficients $h_m^{l_2}(z_1)$. From these three polynomials, we can construct two resultant matrices, $R_1(\{h_1^i(z_1), h_2^j(z_1)\})$ and $R_2(\{h_2^i(z_1), h_3^j(z_1)\})$. Note that the determinants $det(R_1(\{h_1^i(z_1), h_2^j(z_1)\}))$ and $det(R_2(\{h_2^i(z_1), h_3^j(z_1)\}))$ are two one-variable (z_1) polynomials. Therefore, we can construct another resultant matrix $R_3(\{h_m(l_1, l_2)\})$ from these two polynomials.

From Theorem 2, if $\{h_m(z_1, z_2)\}$ share a common zero, say (ζ_1, ζ_2), then

$$det(R_1(\{h_1^i(\zeta_1), h_2^j(\zeta_1)\})) = 0,$$
$$det(R_2(\{h_2^i(\zeta_1), h_3^j(\zeta_1)\})) = 0.$$

This implies $det(R_3(\{h_m(l_1, l_2)\})) = 0$. We can regard this last determinant as a $3L_1L_2$-variable polynomial. From Definition 1, we can define an algebraic set V as the zero set of $\det(R_3(\{h_m(l_1, l_2)\}))$ in the coefficient space $\mathbb{C}^{3L_1L_2}$. Let U be the complement of V. Every element in U satisfies $det(R_3(\{h_m(l_1, l_2)\})) \neq 0$. In other words, the three two-variable polynomials constructed from an element of U share no common zero. From Definition 2, if $dim(V) <$

$dim(\mathbb{C}^{3L_1L_2})$, then U is generic. We know that $dim(V) < dim(\mathbb{C}^{3L_1L_2})$ if $V \neq \mathbb{C}^{3L_1L_2}$ from Theorem 1. Therefore, if we can find a point $\mathbf{x} \in \mathbb{C}^{3L_1L_2} - V$, we will have finished the proof that condition 1 holds generically.

Define three polynomials:

$$
\begin{aligned}
g_1(z_1, z_2) &= z_1^{L_1-1} z_2^{L_2-1} \,, \\
g_2(z_1, z_2) &= z_1^{L_1-1} z_2^{L_2-1} + 2z_1^{L_1-1} + 2z_2^{L_2-1} \,, \\
g_3(z_1, z_2) &= z_1^{L_1-1} z_2^{L_2-1} + z_1^{L_1-1} + z_2^{L_2-1} + 1 \,,
\end{aligned}
$$

and let $g_m^i(z_1)$ be defined analogously to $h_m^i(z_1)$ in (1). Then, we can obtain two polynomials

$$
\begin{aligned}
det(R_1(\{g_1^i(z_1), g_2^j(z_1)\})) &= \left((z_1^{L_1-1})(2z_1^{L_1-1})\right)^{L_2-2} \,, \\
det(R_2(\{g_2^i(z_1), g_3^j(z_1)\})) &= \left((2 - z_1^{L_1-1})(z_1^{L_1-1}+1)\right)^{L_2-2} \,.
\end{aligned}
$$

Because these two polynomials share no common zero and the leading element of $R_3(\{g_m(l_1, l_2)\})$ is not zero, we obtain $det(R_3(\{h_m(l_1, l_2)\})) \neq 0$. This means we have found a point $\mathbf{x} \in \mathbb{C}^{3L_1L_2} - V$.

By following the same argument as above, we can prove that each of conditions 2-4 holds generically. From the Remark, we conclude that the conjunction of conditions 1-4 holds generically if $M \geq 3$. □

Lemma 2: If $N_1 \geq 2r_1+1$ and $N_2 \geq 2r_2+1$, then condition 5 holds generically.
Proof: First, we want to prove that, generically, $\mathbf{S}(r_1+1, r_2)$ has full row rank if $N_1 \geq 2r_1+1$ and $N_2 \geq 2r_2-1$.

Observe that there are only N_1N_2 independent elements in $\mathbf{S}(r_1+1, r_2)$. Let T_i, $i = 1, 2, \ldots, t$, be the $(r_1+1)r_2 \times (r_1+1)r_2$ minors of $\mathbf{S}(r_1+1, r_2)$. Here,

$$
t = \begin{pmatrix} (N_1 - r_1)(N_2 - r_2 + 1) \\ (r_1+1)r_2 \end{pmatrix} .
$$

Let V be the algebraic subset of $\mathbb{C}^{N_1N_2}$ that is the set of common zeros of $\{\det(T_i) \colon i = 1, \ldots, t\}$. We know that $\mathbf{S}(r_1+1, r_2)$ has full column rank if and only if $s \notin V$. Following the same argument as in the proof of Lemma 1, if we can find a point $(\in \mathbb{C}^{N_1N_2})$ which is not in V, then we will have finished the proof that $\mathbf{S}(r_1+1, r_2)$ generically has full row rank.

Assume that $N_1 \geq 2r_1+1$ and $N_2 \geq 2r_2-1$ and choose

$$
s(n_1, n_2) = \begin{cases} 1 & \text{if } n_1 = N_1 - r_1 \text{ and } n_2 = r_2 \\ 0 & \text{else} \end{cases}
$$

Then $\mathbf{S}(r_1+1, r_2)$ will be equal to

$$
r_1+1 \text{ blocks} \left\{ \underbrace{\begin{bmatrix}
\mathbf{0} & \mathbf{0} & \cdots & \cdots & \cdots & \mathbf{0} & S \\
\mathbf{0} & \iddots & \iddots & \iddots & \mathbf{0} & S & \mathbf{0} \\
\vdots & \iddots & \iddots & \iddots & \iddots & \iddots & \vdots \\
\mathbf{0} & \cdots & \mathbf{0} & S & \mathbf{0} & \cdots & \mathbf{0}
\end{bmatrix}}_{N_1 - r_1 \text{ blocks}} \right.
$$

where

$$
S = \begin{bmatrix}
0 & \cdots & 0 & 1 & 0 & \cdots & 0 \\
\vdots & \iddots & 1 & 0 & \iddots & \iddots & 0 \\
\vdots & \iddots & \iddots & \iddots & \iddots & \iddots & \vdots \\
1 & 0 & \cdots & \cdots & \cdots & \cdots & 0
\end{bmatrix}
$$

is an $r_2 \times (N_2 - r_2 + 1)$ matrix. Therefore, $\mathbf{S}(r_1+1, r_2)$ has full row rank.

By following the same argument as above, we can prove that if $N_1 \geq 2r_1 - 1$ and $N_2 \geq 2r_2 + 1$, then $\mathbf{S}(r_1, r_2+1)$ generically has full row rank. Therefore, from the Remark, we have finished the proof. □

Theorem 4: Generically, if $M \geq 3, N_1 \geq 2r_1+1$, and $N_2 \geq 2r_2+1$, then the original image s can be exactly restored, up to a scalar ambiguity, by choosing $K_1 > (L_1-1)r_2$ and $K_2 > L_2 - 1$.

5. EXTENSION TO HIGHER DIMENSIONS

In [3], we proved the sufficient conditions for exact image restoration by using the *generalized Sylvester matrix theorem* [9]. The main technique, like the one in the proof of Lemma 1, is to reduce one variable at a time. By using this technique, we can rewrite conditions 1-4 for exact multichannel blind image restoration as the following conditions for exact multichannel blind n-dimensional signal restoration:

1. The polynomials

   $$
   h_m(z_1 \ldots, z_n) = \sum_{l_1=0}^{L_1-1} \cdots \sum_{l_n=0}^{L_n-1} h_m(L_1 - l_1 - 1, \ldots, L_n - l_n - 1) z_1^{l_1} \ldots z_n^{l_n}
   $$

   for $m = 1, 2, \ldots, M$ share no common zero;
2. Other less-than-n-variable polynomial sets with coefficients from $h_m(l_1, \ldots, l_n)$ share no common zero.

Therefore, by following the same argument as we made in Section 4, we can prove these two conditions hold generically given $M \geq n+1$. Similarly, we can rewrite condition 5 and prove that it holds generically.

Theorem 5: Generically, if $M \geq n+1, N_1 \geq 2r_1+1, \ldots,$ and $N_n \geq 2r_n+1$, then the original n-dimensional signal can be exactly restored, up to a scalar ambiguity, by choosing

$$
K_i > (L_i - 1) \prod_{j=i+1}^{n} r_j, i = 1, 2, \ldots, n \,.
$$

6. SIMULATION RESULTS

In Figure 2, we present simulation results using 3×5 blur functions and 3 blurred versions of the original image.

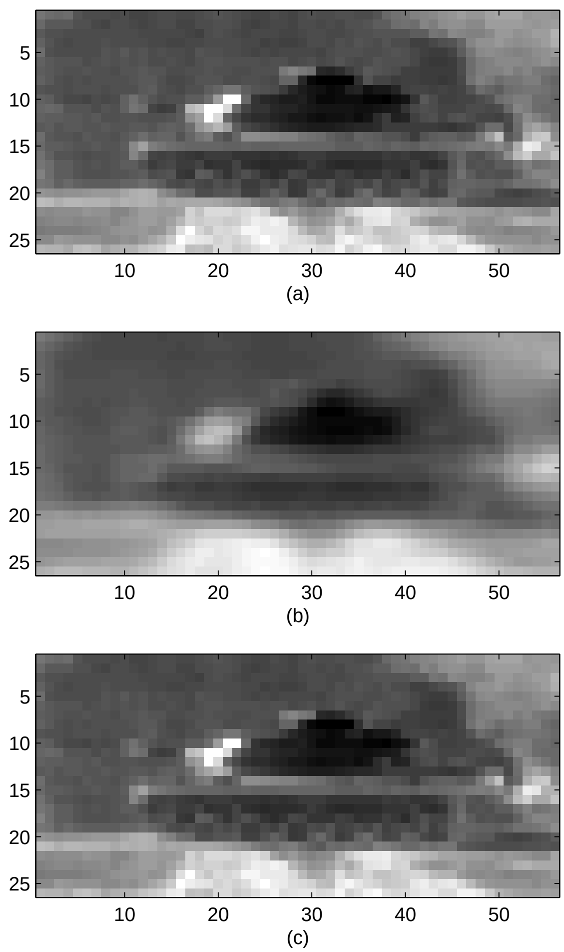

Figure 2: 3×5 blur functions, and 3 channels: (a) the original image, (b) one of the blurred images, and (c) the restored image.

7. CONCLUSIONS AND FUTURE WORK

In this paper, we use the resultant matrix theorem and techniques of algebraic geometry to prove the sufficient conditions for exact multichannel blind image restoration hold generically, given three blurred versions of the same images and some restrictions on the size of the original image. Moreover, our methods can be easily extended to multichannel blind n-dimensional signal restoration. So far, we have developed a solid mathematical foundation. In the future, we will use efficient matrix computation techniques to reduce the computational cost. Also, we will apply optimization techniques to accomplish blind image restoration in the noisy case.

8. REFERENCES

[1] H. Liu and G. Xu, "Closed-form blind symbol estimation in digital communications," *IEEE Trans. on Signal Processing*, vol. 43, pp. 2714–2723, Nov. 1995.

[2] H. T. Pai and A. C. Bovik, "Exact multichannel blind image restoration," *Signal Processing Letters*, vol. 4, pp. 217–220, Aug. 1997.

[3] H. T. Pai, A. C. Bovik, and B. L. Evans, "Multichannel blind image restoration," accepted for publication by *Elektrik*, Aug. 1997.

[4] G. Harikumar and Y. Bresler, "Exact image deconvolution from multiple FIR blurs," submitted to *IEEE Tran. on Image Processing*, 1996.

[5] R. Hartshorne, *Algebraic Geometry.* New York: Springer-Verlag, 1977.

[6] P. Griffiths and J. Harris, *Principles of Algebraic Geometry.* New York: John Wiley & Sons, 1978.

[7] I. R. Shafarevich, *Basic Algebraic Geometry.* New York: Springer-Verlag, 1977.

[8] T. Kailath, *Linear Systems.* Englewood Cliffs, NJ: Prentice-Hall, 1980.

[9] R. R. Bitmead, S. Y. Kung, B. D. O. Anderson, and T. Kailath, "Greatest common divisors via generalized Sylvester and Bezout matrices," *IEEE Trans. on Automatic Control*, vol. 23, pp. 1043–1047, Dec. 1978.