NONRECURSIVE SYNTHESIS OF FIR FILTERS FOR APPROXIMATE PROCESSING*

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ABSTRACT

In approximate and real time processing, one encounters the problem of making efficient use of the limited processing power available. Furthermore it can be desirable to enable a system to react dynamically to a change in requirements. For digital filters this implies the possibility of calculating filter coefficients on the fly with low complexity algorithms. Such an algorithm is presented for the design of linear phase FIR lowpass filters. It has the additional property that subsets of coefficients of one filter constitute by themselves filters of reduced stop band attenuation and/or lower bandwidth reduction.

1. INTRODUCTION

FIR filters play a key role in digital signal processing, due to their stability and the ease of obtaining a linear phase transfer characteristic. Current design algorithms, most often based on the Remez exchange algorithm and its efficient implementation by Parks-McClellan [2], do not give a direct relationship between the desired performance of the filter and its implementation complexity. Furthermore, optimization with respect to fixed point arithmetic has to be done in a computational step separate from the filter design algorithm itself. This is due to the fact that no analytical formula is known, enabling the direct synthesis of efficient FIR lowpass filters. Hence, one has to rely on lengthy iterative calculations with all their implications regarding numerical stability especially for high order filters.

Recently, a second problem has received increasing interest. With the need for efficient use of resources, such as power and time, in highly integrated mobile devices, there is a growing number of applications for algorithms offering the possibility of adjusting their performance with respect to the availability of resources. In approximate signal processing, algorithms are investigated that provide successively better approximations to the desired solution [2].

In this paper, we introduce a new design algorithm which is derived by sampling a weighted sum of eigenfunctions of the Fourier transform. It has the virtue that the basic filter characteristics of a lowpass filter, such as stop band attenuation, bandwidth and slope of the transfer function in the transition band can be influenced directly and independently of each other by choosing the proper set of design parameters. The new filters have the additional property that subsets of filter coefficients of a long filter form filters of relaxed constraints, thus opening the door for easy to design approximate processing. Furthermore the design complexity of the filter is limited to calculating a predefined continuous function in the time domain at the sampling points representing the filter coefficients.

2. NOTATION

In this study we will start by defining a desirable continuous filter function in the frequency domain that will subsequently be transformed to the time domain by inverse Fourier transform. We define the Fourier transform $\mathcal{F}$ of a function f(t) to be F(ω),as given by (1) and its inverse transform as given by (2).

$$\mathcal{F}\{f(t)\} = F(\omega) := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) e^{-j\omega t} dt \qquad (1)$$

$$f(t) = \mathcal{F}^{-1}\{F(\omega)\} := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(\omega) e^{j\omega t} d\omega \qquad (2)$$

This definition of the Fourier transform ensures it is unitary.

By defining the problem of lowpass filtering using only eigenfunctions of the Fourier transforms the question of whether to define a filter in the time or frequency domain

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becomes irrelevant, as every solution in one of the two domains can be readily transposed to the other by multiplying the weighting factor of each eigenfunction with its eigenvalue. For an algorithm defining parameters in the frequency domain, sampling the frequency response and subsequent application of the inverse digital Fourier transform can be replaced by sampling in the time domain.

The eigenfunctions of the continuous Fourier transform are the so called Hermitian functions. They are defined as solutions of the differential equation (3) [4].

$$y''(t) + (2n+1-t^2)y(t) = 0 \tag{3}$$

Recalling the two transform rules (4) and (5)

$$t^n y(t) \Longleftrightarrow j^n Y^{(n)}(\omega) \tag{4}$$

$$y^{(n)}(t) \Longleftrightarrow (j\omega)^n Y(\omega) \tag{5}$$

we can apply the operator Fourier transform to equation (3) and obtain the differential equation (6) which is identical in structure.

$$Y''(\omega) + (2n+1-\omega^2)Y(\omega) = 0 \tag{6}$$

Therefore equation (3) is invariant to the Fourier transform and its solutions, the Hermitian functions, are proven to be the eigenfunctions of the Fourier transform.

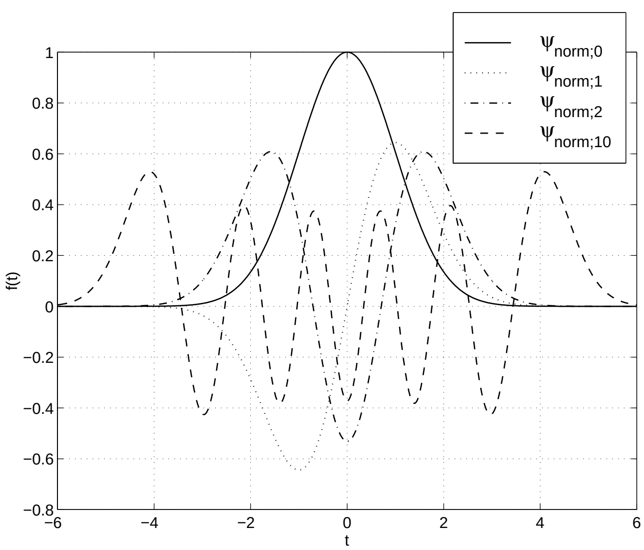

Figure 1. Hermitian functions $\psi_{norm;n}(t) = \psi_n(t)/\sqrt{2^n n! \sqrt{\pi}}$

The Hermitian functions are defined as follows [4] :

$$\psi_n(t) := (-1)^n e^{\frac{t^2}{2}} \frac{d^n}{dt^n}\left[e^{-t^2}\right] \tag{7}$$

$$\Psi_n(\omega) := \mathcal{F}\{\psi_n(t)\} = (-j)^n \psi_n(\omega) \tag{8}$$

ψ_0 being the gaussian pulse. They constitute a complete orthogonal set in $L_2^C(-\infty,\infty)$, which is particularly well suited for the description of lowpass signals, as can be seen in figure 1. Note that close to the origin these functions can be approximated by a cosine function [4] (n>>1) :

$$\psi_{2n}(t) \cong 2^{-n}(2n-1)!!\cos\left(t\sqrt{4n+1}\right) \tag{9}$$

(where $(2n-1)!! = (2n-1)\cdot(2n-3)\cdot\cdots\cdot 3\cdot 1$)

An important property of the eigenfunctions is their very low time-bandwidth product BT. With

$$B = \sqrt{\frac{1}{E}\int_{-\infty}^{\infty}\omega^2\left|Y(\omega)\right|^2 d\omega} \quad \text{and} \quad T = \sqrt{\frac{1}{E}\int_{-\infty}^{\infty} t^2 y^2(t)dt} \tag{10}$$

$$\text{where} \quad E = \int_{-\infty}^{\infty} y^2(t)dt = \int_{-\infty}^{\infty}\left|Y(\omega)\right|^2 d\omega \tag{11}$$

It is well known that all functions verify the Heisenberg uncertainty principle

$$BT \geq \frac{1}{2} \tag{12}$$

with equality holding only for the gaussian pulse ψ_0.

3. DESIGN ALGORITHM

3.1. Continuous function

Being interested only in functions with a lowpass characteristic, we can limit ourselves to eigenfunctions of even degree. Taking the weighted sum of the M even eigenfunctions of lowest degree (0,2,4,...,2(M-1)), H(ω) describes the sought continuous function, where M is the first free parameter of the design algorithm.

$$H(\omega) := \sum_{i=0}^{M-1} x_i \Psi_{2i}(\omega) \tag{13}$$

In order to obtain a lowpass, one can force to zero the first 2M-1 derivatives of H(ω) at the origin (ω=0), noting that the derivatives of odd degree are all zero at the origin regardless of the weights x_i chosen.

$$\left.\frac{d^{(2m)}}{d\omega^{2m}}\left(H(\omega)\right)\right|_{\omega=0} := H^{(2m)}(0) \stackrel{!}{=} 0 \quad \text{for m = 1,2,...,M-1} \tag{14}$$

Setting the gain at the origin to α this results in the following system of linear equations :

$$\begin{pmatrix} \Psi_0(0) & \Psi_2(0) & \cdots & \Psi_{2(M-1)}(0) \\ \Psi_0^{(2)}(0) & \Psi_2^{(2)}(0) & \cdots & \Psi_{2(M-1)}^{(2)}(0) \\ \Psi_0^{(4)}(0) & \Psi_2^{(4)}(0) & \cdots & \Psi_{2(M-1)}^{(4)}(0) \\ \vdots & \vdots & \ddots & \vdots \\ \Psi_0^{(2(M-1))}(0) & \Psi_2^{(2(M-1))}(0) & \cdots & \Psi_{2(M-1)}^{(2(M-1))}(0) \end{pmatrix} \begin{pmatrix} x_0 \\ x_1 \\ x_2 \\ \vdots \\ x_{M-1} \end{pmatrix} = \begin{pmatrix} \alpha \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \quad (15)$$

where

$$\Psi_{2n}^{(2m)}(0) = (-1)^{n+m} 2^n (2n-1)!!(2m-1)!! \sum_{k=0}^{\min(m,n)} \binom{m}{k}\binom{n}{k} 2^{2k} \frac{k!}{(2k-1)!!} \quad (16)$$

Solving this system of linear equations (15), one obtains $H(\omega)$ as

$$H(\omega) = \alpha \cdot e^{\frac{-\omega^2}{2}} \sum_{k=0}^{M-1} \frac{\omega^{2k}}{(2k)!!} \quad (17)$$

Note that this solution can be verified by expanding $\exp(-\omega^2/2)$. By applying the inverse Fourier transform to equation (17) we get the continuous function with lowpass characteristics in the time domain :

$$\boxed{h(t) = \alpha \cdot e^{\frac{-t^2}{2}} \sum_{k=0}^{M-1} (-1)^k \frac{\binom{M-1}{k}}{(2k+1)!!} t^{2k}} \quad (18)$$

Figures 2 and 3 give examples for M=5 and M=15 respectively on a logarithmic scale.

Note that the zeros of the polynomial part of h(t) are approximately equally spaced on the time axis, the first being at $\pm t_{0;M}$.

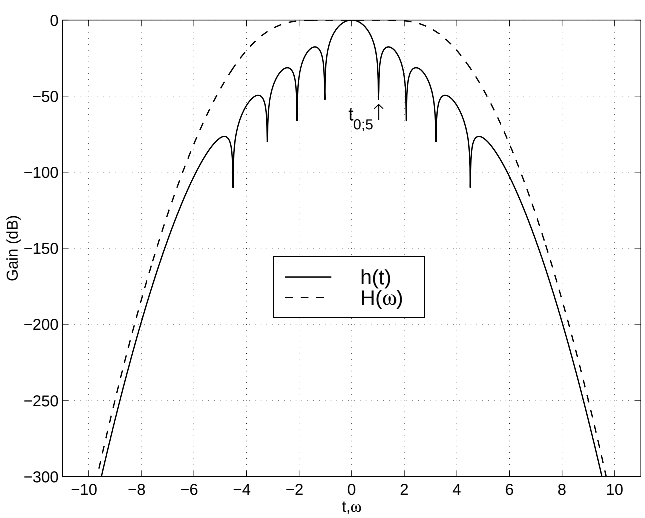

Figure 2. Continuous functions for M = 5

3.2. Conversion to FIR

In the next step the coefficients of the FIR lowpass filter are obtained by sampling the continuous impulse response h(t), within a rectangular window W, at points equally spaced in Δt. In contrast to sampling a desired frequency response and subsequently applying the inverse Fourier transform, the new method has the important advantage that the inverse Fourier transform is replaced by a simple multiplication of the weights of the eigenfunctions with their eigenvalues. As only eigenfunctions up to a chosen degree are used, one is sure that the inverse transform does not introduce unwanted high frequency components.

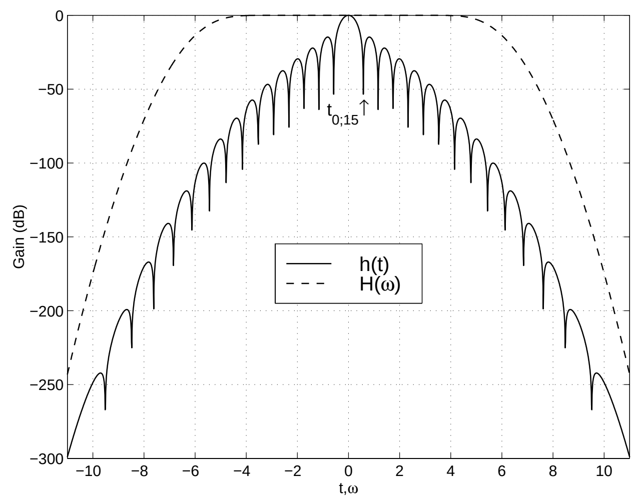

Figure 3. Continuous functions for M=15

As h(t) and H(ω) fall off exponentially in both the time and frequency domain, and additionally all their derivatives exist and are of finite magnitude this simple method of sampling h(t) ensures good results. As can be expected by the lemma of Riemann-Lebesgue [1] there are no problems such as Gibb's phenomenon and the oscillations in the pass-band are very limited due to the fact of H(ω) being maximally flat close to the origin.

As will be shown by examples below, the three major characteristics of an FIR can be set independently of each other with this technique :

- normalized bandwidth B := B@6dB = $\Delta t/t_0$ (cf. fig. 4)
- attenuation A : increasing with W (cf. fig. 5)
- shape factor SF:=B@6dB/B@60dB : increasing with M (cf. fig. 6)

Together, these three choices set the filter size N.

4. EXAMPLES

The examples given in figures 4 to 6 not only demonstrate the properties stated above but they also show the behavior of the pass band ripple.

Firstly we find that a reduction in bandwidth, forced by a decrease in sampling step size does not influence the magnitude of the pass band ripple, cf. figure 4.

Secondly figures 5 demonstrates the decrease in ripple caused by a decrease in truncation noise due to an increase in the size of the sampling window.

Finally, having chosen a maximally flat continuous function it is not surprising that the ripple diminishes with an increase in flatness induced by an augmented number of eigenfunctions used, as it is shown by figure 6.

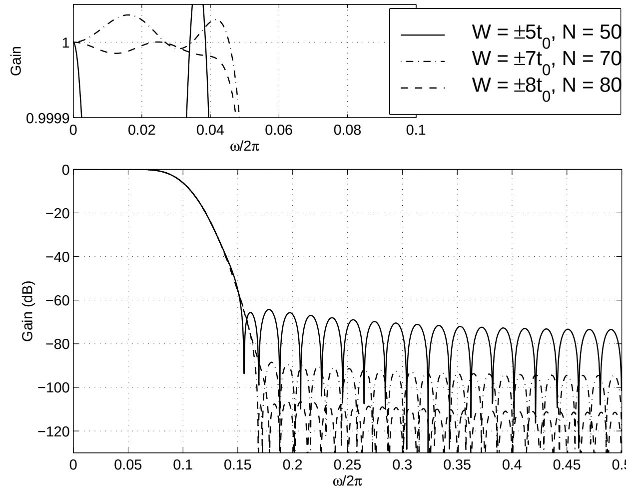

Figure 5. Influence of W (M=10, 1/B=5)

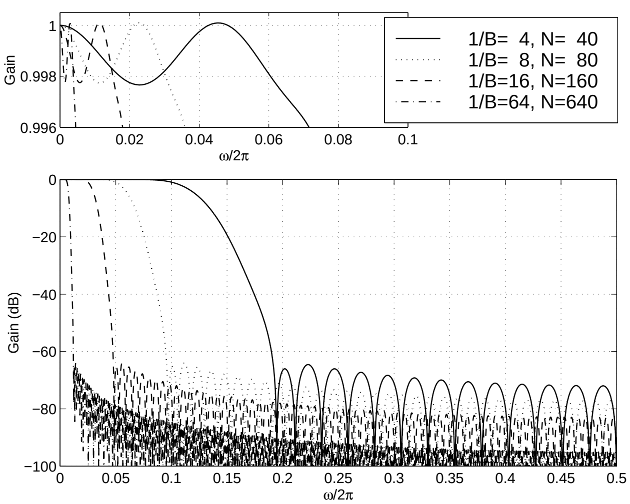

Figure 4. Influence of Δt (M=10, W=$\pm 5t_0$)

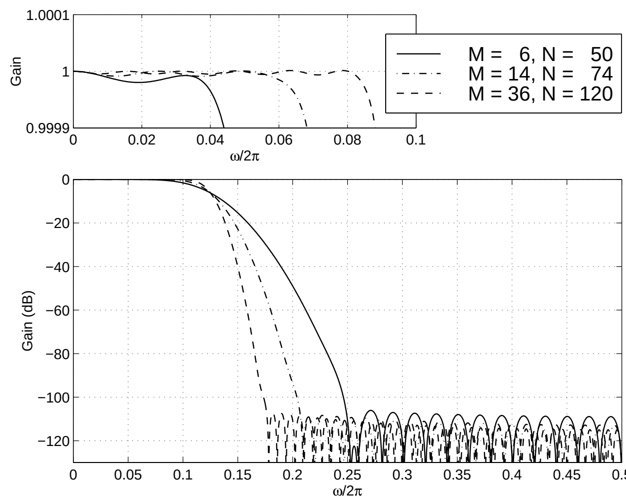

Figure 6. Influence of number of eigenfunctions (1/B=4)

5. SUMMARY

A new method for the design of FIR lowpass filters has been described. It has the virtue of very low design complexity and subsets of filter coefficients form filters satisfying relaxed constraints. The mathematical description of the algorithm makes the design complexity virtually independent of the filters size. Hence, filters of any quality with respect to bandwidth, stopband attenuation and transition band size can be derived. Furthermore, it will be interesting to see whether the mathematical description makes it possible to obtain important savings in implementation complexity by transforming the direct FIR structure onto new to be developed ones.

6. REFERENCES

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