WAVELET-VAGUELETTE RESTORATION IN PHOTON-LIMITED IMAGING

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ABSTRACT

This paper studies linear shift-invariant inverse problems arising in photon-limited imaging. The problem we consider is the recovery of an intensity image from a distorted version degraded with Poisson noise. This problem arises in medical and astronomical imaging. It is shown that the wavelet-vaguelette decomposition (WVD) can provide much better estimates of the underlying intensity compared to classical frequency domain methods. The paper combines recently developed wavelet-based filtering techniques for photon imaging with new results in WVD methods for inverse problems. Furthermore, we show that the WVD can be interpreted as a prefiltered wavelet transform, and that it can be very efficiently computed. The new method is applied to nuclear medicine imaging.

1. INTRODUCTION

1.1. LSI Inverse Problems in Photon-Limited Imaging

Linear shift-invariant (LSI) inverse problems arise in many imaging applications. For example, in nuclear medicine imaging, camera collimators introduce a distortion that is typically modeled as LSI [1, pp. 157-162]. Similar problems arise in many optical imaging systems [2]. The general form of the discrete inverse problem considered in this paper is

$$\mathbf{c} \sim \text{Poisson}\{\mathbf{K}\,\boldsymbol{\lambda}\} \tag{1}$$

where $\mathbf{c}$ is the $2^M \times 2^M$ observed count image whose distribution is Poisson with intensity $\mathbf{K}\,\boldsymbol{\lambda}$. $\mathbf{K}$ is a known circular convolution operator, $\boldsymbol{\lambda}$ is the unknown intensity image of interest which we also regard as a discrete $2^M \times 2^M$ image. All quantities are real-valued. The objective is to recover the $\boldsymbol{\lambda}$ from the counts $\mathbf{c}$. Throughout the paper, $N = (2^M)^2$ is the number of pixels in the image.

Many methods have been proposed for recovering $\boldsymbol{\lambda}$ from $\mathbf{c}$ (*e.g.*, [2, 3]). In this paper, we develop a new method based on two recent developments in wavelet-based signal estimation; the wavelet-vaguelette decomposition (WVD) [4] and wavelet-domain filters for photon imaging [5]. Previously proposed WVD methods were derived under the assumption of Gaussian noise [4, 6]. We develop a new WVD method specifically designed to handle Poisson noise. We apply the new method to nuclear medicine imaging, and compare its performance to that of an existing method routinely used in practice.

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1.2. Classical Frequency-Domain Solutions

One solution to the inverse problem is to take $\mathbf{c}$ as an estimate of $\mathbf{K}\,\boldsymbol{\lambda}$, and then, assuming $\mathbf{K}$ is invertible, form the estimate of $\boldsymbol{\lambda}$

$$\widehat{\boldsymbol{\lambda}} = \mathbf{K}^{-1}\mathbf{c} \tag{2}$$

Note that $\mathbf{c}$ is an unbiased estimator of $\mathbf{K}\,\boldsymbol{\lambda}$, and therefore $\widehat{\boldsymbol{\lambda}}$ is an unbiased estimator of $\boldsymbol{\lambda}$. However, the problem with this solution is that the inversion may result in excessive noise amplification and unacceptable estimator variance.

The reason for this is easily seen by expanding the operator $\mathbf{K}$ in its eigendecomposition. Since $\mathbf{K}$ is a circular convolution operator its eigenvectors $\{\mathbf{e}_n\}$ are the 2-d discrete Fourier transform (DFT) vectors. Let $\{\mu_n\}$ denote the eigenvalues of $\mathbf{K}$. Expanding (2) in terms of the eigendecomposition of $\mathbf{K}$ we have

$$\widehat{\boldsymbol{\lambda}} = \sum_n \mu_n^{-1}\,\langle \mathbf{e}_n, \mathbf{c}\rangle\,\mathbf{e}_n \tag{3}$$

where $\langle \mathbf{e}_n, \mathbf{c}\rangle$ denotes the inner product between $\mathbf{c}$ and $\mathbf{e}_n$.

The variance of each term in the expansion is computed as follows. Because the data are Poisson distributed and since the magnitude of each element in the DFT vector is a constant $(1/\sqrt{N})$, the variance of each inner product $\langle \mathbf{e}_n, \mathbf{c}\rangle$ is a constant σ^2 that is proportional to the sum of all elements in $\mathbf{K}\boldsymbol{\lambda}$ [5]. Therefore the variance of the n-th term in (3) is $\mu_n^{-2}\sigma^2$. This shows that the Poisson noise is greatly amplified in the components of the solution associated with small eigenvalues of the operator $\mathbf{K}$.

To reduce noise amplification we can employ a *windowed singular value decomposition (SVD) reconstruction*

$$\bar{\boldsymbol{\lambda}} = \sum_{n=0}^{N-1} \omega_n \mu_n^{-1}\,\langle \mathbf{e}_n, \mathbf{c}\rangle\,\mathbf{e}_n \tag{4}$$

The weights $\{\omega_n\}$ are chosen to reduce the amplification of noise introduced by $\{\mu_n^{-1}\}$. Since we are dealing with circular convolution, in this case the windowed SVD method is a DFT-domain (frequency-domain) filter. Many filters of this form have been proposed in photon-limited imaging. In particular, the *Metz filter* is commonly used in nuclear medicine imaging [3]. Another common filter is the *truncated SVD* solution which is obtained by setting weights below a cut-off frequency to 1 and above the cut-off to 0 [6].

1.3. Drawbacks of Frequency-Domain Method

The key limitation of the frequency-domain method is that it matches the reconstruction basis to the operator $\mathbf{K}$ with no regard for the underlying signal of interest. The frequency-domain method uses the DFT vectors which have spatial support over the entire signal. Hence the frequency-domain method can not adjust to spatial variations in the behavior of the signal. Recently it has been shown that much better solutions can be obtained by matching the basis functions to the signal rather than to the operator [4, 6]. However, we emphasize that previous work dealt exclusively with Gaussian noise contamination.

This paper considers a discrete version of Donoho's wavelet-vaguelette decomposition (WVD) method for linear inverse problems [4]. It has been shown that the WVD solution can attain optimal convergence rates [4]. In Section 2, we review wavelet-domain filtering for direct photon-limited imaging. In Section 3, we derive a new WVD algorithm for image restoration problems involving Poisson noise. In Section 4, we develop an efficient algorithm for computing the WVD restoration. In Section 5, we apply the algorithm to nuclear medicine imaging, and it is shown that the WVD method can provide much better solutions than frequency-domain techniques. In Section 6, we make concluding remarks and indicate possible directions for future work.

2. WAVELET-DOMAIN FILTERING FOR PHOTON-LIMITED IMAGING

The discrete wavelet transform (DWT) provides a very concise representation for wide classes of real-world signals. This property has been exploited to develop extremely low-bit-rate compression algorithms [7] and powerful signal denoising and estimation methods [4]. In this section, we review the DWT and wavelet-domain filtering for noise removal in direct photon-limited imaging.

2.1. The Discrete Wavelet Transform

The 2-d orthogonal DWT represents a real-valued, $2^M \times 2^M$ image in terms of shifts and dilations of a lowpass *scaling function* and bandpass *wavelet functions* [7]. The scaling and wavelet coefficients can be easily computed using a 2-d filter bank consisting of lowpass and highpass filters and decimators [7, pp.269-270]. Due to the special filter bank structure, the forward and inverse DWT can be computed in $O(N)$ operations. *Throughout the paper we restrict our attention to the periodic DWT.* This means that all convolutions in the filter bank are circular. The underlying 2-d wavelet functions at each scale have three distinct orientations — horizontal, vertical, and diagonal. Together the wavelet and scaling functions provide an orthonormal basis for 2-d images. For more details on 2-d wavelet transforms see [7].

To simplify notation, we denote the underlying wavelet and scaling functions collectively by the abstract notation $\{\mathbf{w}_n\}_{n=0}^{N-1}$. Thus, an image $\boldsymbol{\lambda}$ can be represented in terms of the DWT as

$$\boldsymbol{\lambda} = \sum_{n=0}^{N-1} \theta_n \mathbf{w}_n \tag{5}$$

where the DWT coefficients $\theta_n = \langle \mathbf{w}_n, \boldsymbol{\lambda} \rangle$, the inner product between the basis function and the image.

2.2. Wavelet-Domain Filtering for Direct Photon Imaging

Wavelet-domain filtering for direct photon imaging ($\mathbf{K} = \text{Identity}$) is studied extensively in [5, 8]. In this paper, we develop an image restoration algorithm based on the approach proposed in [5] which is described as follows. Given direct counts $\mathbf{c} \sim \text{Poisson}\{\boldsymbol{\lambda}\}$, compute the DWT of $\mathbf{c}$ and let $\{\widehat{\theta}_n\}$ denote the DWT coefficients. Next, filter the raw DWT coefficients $\widehat{\theta}_n$ to obtain new coefficients $\widetilde{\theta}_n$ according to

$$\widetilde{\theta}_n = \left(\frac{\widehat{\theta}_n^2 - \alpha\, \sigma_n^2}{\widehat{\theta}_n^2} \right)_+ \widehat{\theta}_n \tag{6}$$

where σ_n^2 is an unbiased estimator of the noise power in $\widehat{\theta}_n$, $\alpha \geq 0$ is a gain factor, and $(\cdot)_+$ denotes the positive part of the argument. The mapping $\widehat{\theta}_n \mapsto \widetilde{\theta}_n$ is a nonlinear threshold operation — coefficients with low signal-to-noise ratio (SNR) are set to zero, and those with high SNR are left essentially unaltered. Thus, the filter removes small wavelet coefficients that contain significantly more noise than signal. Increasing the gain α increases the threshold. The filtered image is reconstructed by computing the inverse DWT of the filtered coefficients $\{\widetilde{\theta}_n\}$. It can be shown that if $\alpha = 1$, then this filtering procedure minimizes a predictive sum of squared errors, and is asymptotically optimal in the mean square error sense [5, 9].

3. WVD METHODS FOR INVERSE PROBLEMS IN PHOTON-LIMITED IMAGING

The problem we are interested in is recovering an intensity image $\boldsymbol{\lambda}$ from observed counts $\mathbf{c} \sim \text{Poisson}\{\mathbf{K}\boldsymbol{\lambda}\}$. The idea of the WVD method is to use the wavelet functions, rather than the eigenfunctions of the operator $\mathbf{K}$, in the inversion process since the wavelet transform provides an efficient representation for many signals. Hence, the WVD method matches the basis functions to the signal rather than the operator. Furthermore, because the wavelet functions are localized in both space and frequency the WVD enables a spatially adaptive inversion. In this section, we use the WVD to extend the wavelet-domain filtering techniques described above to the inverse problem at hand.

3.1. The Wavelet-Vaguelette Decomposition

The goal of the WVD is to express the solution to the inverse problem in terms of the wavelet and scaling functions. Equation (5) shows that all we need to reconstruct the signal are the DWT coefficients of $\boldsymbol{\lambda}$. The WVD can be used to obtain unbiased estimates of the DWT coefficients from the indirect data $\mathbf{c} \sim \text{Poisson}\{\mathbf{K}\boldsymbol{\lambda}\}$. Let $\mathbf{K}^{-*}$ denote the adjoint of $\mathbf{K}^{-1}$, and define $\mathbf{u}_n = \mathbf{K}^{-*}\mathbf{w}_n$. Similar to the terminology in [4], $\mathbf{u}_n$ is called a *vaguelette*. Consider the inner product between the data and the vaguelette

$$\widehat{\theta}_n = \langle \mathbf{u}_n, \mathbf{c} \rangle = \left\langle \mathbf{K}^{-*}\mathbf{w}_n, \mathbf{c} \right\rangle \tag{7}$$

The expected value of $\widehat{\theta}_n$ is

$$\mathrm{E}\left[\widehat{\theta}_n\right] = \left\langle \mathbf{K}^{-*}\mathbf{w}_n, \mathbf{K}\,\boldsymbol{\lambda} \right\rangle = \langle \mathbf{w}_n, \boldsymbol{\lambda} \rangle \tag{8}$$

where we exploit the linearity of the inner product and definition of $\mathbf{K}^{-*}$.[1] Hence, the vaguelette coefficients are unbiased estimators of the DWT coefficients of $\boldsymbol{\lambda}$.

The vaguelette coefficients of the data are noisy versions of the DWT coefficients of the signal we are trying to recover. Plugging the vaguelette coefficients directly into (5) in place of the true

[1] $\left\langle \mathbf{K}^{-*}\mathbf{w}_n, \mathbf{K}\,\boldsymbol{\lambda} \right\rangle = \left\langle \mathbf{w}_n, \mathbf{K}^{-1}\mathbf{K}\,\boldsymbol{\lambda} \right\rangle = \langle \mathbf{w}_n, \boldsymbol{\lambda} \rangle.$

DWT coefficients will only reproduce (2). To improve the solution we must modify the vaguelette coefficients prior to reconstruction.

3.2. WVD Methods for Photon-Limited Imaging

In the spirit of the wavelet-domain filtering method for photon imaging outlined in Section 2, we can improve the estimates $\{\widehat{\theta}_n\}$ of $\boldsymbol{\lambda}$'s DWT coefficients by applying the threshold nonlinearity (6) to the vaguelette coefficients. To construct the threshold operation, we require estimators of the noise power in each vaguelette coefficient. Each vaguelette coefficient is a linear combination of Poisson variates. Let us denote this by

$$\widehat{\theta}_n = \langle \mathbf{u}_n, \mathbf{c} \rangle = \sum_i u_{i,n} c_i \tag{9}$$

where $c_i \sim \text{Poisson}(\lambda_i)$. Then the variance of the vaguelette coefficient is

$$\sigma_n^2 = \text{Var}\left(\widehat{\theta}_n\right) = \sum_i u_{i,n}^2 \lambda_i \tag{10}$$

Therefore an unbiased estimator of the variance is

$$\widehat{\sigma_n^2} = \sum_i u_{i,n}^2 c_i \tag{11}$$

In other words, the inner product between the data and the point-wise square of the vaguelette provides an unbiased estimator of the noise power in the corresponding vaguelette coefficient.

With this estimator for the noise power, we can apply the threshold to the vaguelette coefficients:

$$\widetilde{\theta}_n = \left(\frac{\widehat{\theta}_n^2 - \alpha\, \widehat{\sigma_n^2}}{\widehat{\theta}_n^2} \right)_+ \widehat{\theta}_n \tag{12}$$

In effect, the threshold discards all vaguelette coefficients except those that contain significant signal energy. Again, the gain α is chosen by the user. If $\alpha = 1$, then following the analysis in [9] it is easy to show that this threshold minimizes a predictive sum of squared error criterion for the inverse problem. Also, in this case the threshold above is asymptotically optimal in the mean-square-error sense. Larger values of α correspond to more aggressive thresholding and thus more regularization.

The WVD solution to the LSI inverse problem is

$$\widetilde{\boldsymbol{\lambda}} = \sum_{n=0}^{N-1} \widetilde{\theta}_n \mathbf{w}_n \tag{13}$$

$\widetilde{\boldsymbol{\lambda}}$ can be computed by taking the inverse DWT (IDWT) of the thresholded vaguelette coefficients. In this process, it is possible that a pixel intensity estimate is negative. However, in our experience this is not a significant problem — only a few low-intensity pixels may have negative estimates. A simple remedy is to set negative pixel estimates to zero. The next section shows that the WVD solution can be computed very efficiently.

4. COMPUTING THE WVD

The vaguelette coefficients can be obtained directly by forming $\mathbf{K}^{-*}$, generating $\mathbf{u}_n = \mathbf{K}^{-*}\mathbf{w}_n$, and computing the inner products in (7), requiring $O(N^3)$ operations, where N is the number of pixels in the image. The squared vaguelette that is required for the noise power estimator is easily obtained by squaring each element of $\mathbf{u}_n$ and computing its inner product with $\mathbf{c}$. However, these calculations can be computed much more efficiently by exploiting the shift-invariance of the operator $\mathbf{K}$ and the fact that we are working with the periodic DWT.

The vaguelette coefficients at the j-th scale in one of the three orientations (horizontal, vertical, and diagonal) can be computed by circularly convolving $\mathbf{c}$ with $\mathbf{K}^{-*}\mathbf{w}$, where $\mathbf{w}$ is a representative wavelet function at the j-th scale in the proper orientation, and then downsampling by 2^j in both vertical and horizontal directions. Note that since $\mathbf{K}$ is a circular convolution operator so is $\mathbf{K}^{-*}$. Therefore $\mathbf{K}^{-*}\mathbf{w}$ can be computed in $O(NlogN)$ operations.[2] The noise power estimates are computed by convolving $\mathbf{c}$ with the point-wise square of $\mathbf{K}^{-*}\mathbf{w}$ and downsampling. The overall complexity of computing the vaguelette coefficients and noise estimates at the j-th scale is $O(NlogN)$ and we have at most $O(logN)$ scales. Hence, the overall cost of computing the WVD inverse (13) is $O(Nlog^2N)$.

Note that the vaguelette coefficients can be even more efficiently computed. Since $\mathbf{K}$ and the highpass and lowpass filters in the DWT filter bank are all circular convolutions, it is easy to show that the vaguelette coefficients of $\mathbf{c}$ are equal to the DWT coefficients of $\mathbf{K}^{-*}\mathbf{c}$ [6]. Hence, the vaguelette coefficients are simply the coefficients of a prefiltered DWT and therefore can be computed in $O(NlogN)$ operations. Note, however, that the same "trick" can not be used to compute the noise power estimates due to the squaring of the vaguelette functions. Therefore, the overall complexity of the WVD inverse is still $O(Nlog^2N)$.

The fast WVD algorithm is summarized below.

Fast WVD for LSI Inverse Problems in Photon-Limited Imaging

1. Use FFT to compute circular convolution $\mathbf{K}^{-*}\mathbf{c}$
   $O(NlogN)$ *operations*
2. Compute DWT of $\mathbf{K}^{-*}\mathbf{c}$ to obtain vaguelette coefficients
   $O(N)$ *operations*
3. Compute vaguelette noise power estimates
   $O(N\log^2 N)$ *operations*
4. Compute solution by taking IDWT of thresholded vaguelette coefficients
   $O(N)$ *operations*

5. APPLICATION TO NUCLEAR MEDICINE IMAGING

Nuclear medicine images are formed by detecting gamma-ray photons emmitted as a radioactive pharmaceutical decays inside a human patient [1, pp. 157-162]. A nuclear medicine spine image, acquired with a General Electric Starcam System, is shown Figure 1 (a). The gamma-ray emission process is well-modeled by the Poisson distribution. A lead collimator, placed in front of the detectors, produces a significant distortion in the data that is well-modeled as a lowpass LSI filter $\mathbf{K}$ that is radially symmetric in the frequency-domain as depicted in Figure 2. Hence, the counts

[2] The inverse DWT used to compute $\mathbf{w}$ requires $O(N)$ operations and the circular convolution $\mathbf{K}^{-*}\mathbf{w}$ requires $O(NlogN)$ operations.

that are detected can be modeled by (1). The frequency response is severely lowpass, and drastically reduces spatial resolution.

In this application the distortion is real and symmetric, and therefore $\mathbf{K}^{-*} = \mathbf{K}^{-1}$. The inversion process (resolution recovery) is extremely ill-posed, and in practice an approximation to the inverse of $\mathbf{K}$ is used instead of $\mathbf{K}^{-1}$. The classical approach to resolution recovery in nuclear medicine is the Metz filter [3]. Two Metz filters are depicted in Figure 2. The Metz filter attempts to strike a balance between inverse filtering and noise amplification. The Metz filter I restoration of the spine image is shown in Figure 1 (b). More aggressive filtering is possible using Metz filter II, but this can lead to excessive noise amplification (see Figure 1 (c)). However, using the WVD method with Metz filter II to approximate $\mathbf{K}^{-1}$ (see Figure 1 (d)) provides much better resolution recovery than Metz filter I, and eliminates the excessive noise amplification that is incurred in the direct application of Metz filter II. In this example, we used the Daubechies-8 wavelet since, due to its good regularity and localization properties, it should provide an excellent match to the intensity of interest. Also, the gain factor $\alpha = 2$ in (12).

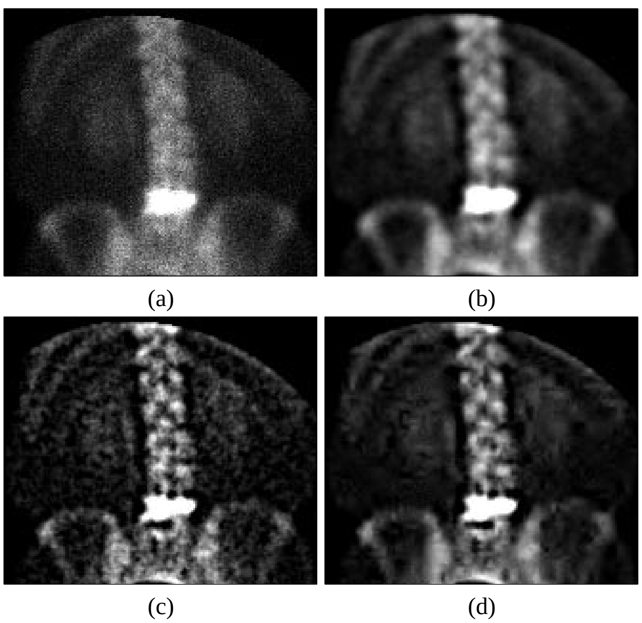

Figure 1: *Restoration of nuclear medicine image. (a) Original noisy and blurred spine image, (b) image restoration using Metz filter I, (c) image restoration using Metz filter II alone, (d) restoration using WVD method with Metz filter II. Metz filter II recovers more resolution than the Metz filter I, but at a cost of greater noise amplification. Using Metz filter II in the WVD algorithm produces a very nice, high resolution image without noise amplification.*

6. CONCLUSIONS

This paper described a novel WVD method for inverse problems in photon-limited imaging. We have applied the new algorithm to nuclear medicine imaging, and we have demonstrated that it offers advantages over classical frequency-domain filtering approaches. In particular, the WVD method enables resolution recovery without degrading noise artifacts such as those generated by the Metz

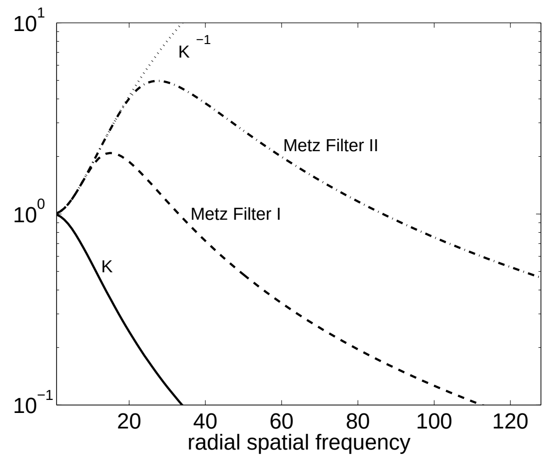

Figure 2: *Inverse filters for nuclear medicine imaging. The frequency response $\mathbf{K}$ of the imaging system, Metz filter I, and Metz filter II are pictured above. For comparison the true inverse $\mathbf{K}^{-1}$ is also shown.*

filter. Furthermore, simulated data studies with Gaussian noise [6] and Poisson noise [10] have shown that the WVD method significantly outperforms frequency-domain filtering techniques like the Metz filter. In future work, we plan to compare the WVD method to Maximum Likelihood-based approaches [2].

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