SHORT-TIME FOURIER ANALYSIS - A NOVEL WINDOW DESIGN PROCEDURE

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ABSTRACT

Weyl-Heisenberg frames are the tool for short-time Fourier analysis. These are generated from a prototype window function using translation on a rectangular grid in the time-frequency plane. Particularly appealing Weyl-Heisenberg frames are those which are tight as they allow for signal representations analogous to orthonormal expansions and have good numerical stability properties. Designing the window of a tight Weyl-Heisenberg frame requires optimization of the frequency characteristics of the window, usually some form of frequency selectivity, under a set of nonlinear constraints. For long windows this can be a formidable task, if not infeasible. We propose a new filter design method based on expansions with respect to prolate spheroidal sequences. The advantages of this new method are more and more pronounced as the redundancy of the frame increases. These advantages pertain to a reduction in computational complexity and the ability to describe good and long windows with a few parameters.

1. INTRODUCTION

Short-time Fourier analysis, as originally proposed by Gabor [7], amounts to expanding signals with respect to a Weyl-Heisenberg family of vectors

$$\Phi_{v,x_0,\omega_0} = \{v_{lm} : v_{lm}(x) = v(x - lx_0)e^{jm\omega_0 x}\}_{l\in\mathbf{Z},m\in\mathbf{Z}}, \quad (1)$$

that are generated by translating a single prototype window function in time and frequency. In digital signal processing, one often encounters representations which are obtained as inner products $\langle f, v_{lm}\rangle$ of a signal $f \in \ell^2(\mathbf{Z})$ with the vectors of a discrete-time Weyl-Heisenberg family

$$\Phi_{v,N,K} = \{v_{lm} : v_{lm}[n] = v[n - mN]e^{j\frac{2\pi}{K}ln}\}_{l\in\mathbf{Z}_K,m\in\mathbf{Z}},^{1} \quad (2)$$

rather than expansions with respect to $\Phi_{v,N,K}$. The transform that maps $f \in \ell^2(\mathbf{Z})$ to the collection of inner products

$$F(l,m) = \langle f, v_{lm}\rangle, \;\; v_{lm} \in \Phi_{v,N,K}$$

is referred to as the short-time Fourier transform and it can be implemented using modulated filter banks. The goal of signal analysis using either expansions

$$f = \sum_{l,m} a_{lm} v_{lm}, \quad (3)$$

with respect to Weyl-Heisenberg families or the inner product representations is to extract information on the spectral content of the signal without sacrificing information on its localization in time and to facilitate signal processing locally in both time and frequency. Hence, it is crucial to deal with windows which are well localized in both time and frequency. It is also important that Φ_v is complete in the considered space ($L^2(\mathbf{R})$ or $\ell^2(\mathbf{Z})$) and exhibit certain stability properties, in the sense that a small perturbation in the expansion coefficients of f cannot result in a signal which is arbitrarily far from f; similarly for the inner products representation. A family of vectors in a space which satisfies these completeness and stability requirements is said to be a *frame* [6, 5]. For its many fine features, the preferable class of frames are orthonormal bases. However, it turns out that the requirements for linear independence of vectors in a frame Φ_v and good localization of the window v are in conflict.

A result known as the Balian-Low theorem [2, 8] asserts that if v is a window of an orthonormal Weyl-Heisenberg basis in $L^2(\mathbf{R})$ then it has slow decay (i.e. poor localization) in either time or frequency. An effect in $\ell^2(\mathbf{Z})$ similar to that described by the Balian-Low theorem has been observed in [11], where it is shown that there are no critically sampled modulated filter banks with finite impulse responses that have good frequency selectivity. This was the motivation for research in the direction of redundant Weyl-Heisenberg frames. It was demonstrated by Daubechies that as soon as some redundancy is introduced the situation changes drastically, that is redundant Weyl-Heisenberg frames in $L^2(\mathbf{R})$ allow for windows with good localization in both time and frequency [5]. It was also shown that in $\ell^2(\mathbf{Z})$, Weyl-Heisenberg families based on windows with good frequency selectivity are attainable when redundancy is allowed [4]. Even before these results on the limitations of Weyl-Heisenberg bases were established, it had been known in the signal processing community that redundant short-time Fourier representations are advantageous over critically sampled ones in terms of providing robustness, which is important in applications involving some processing in the Fourier domain [1].

A particularly interesting class of redundant Weyl-Heisenberg frames are those which are tight. A convenience of of dealing with tight frames is that signals can be expanded in a manner reminiscent of orthonormal expansions. Namely, if $\Phi_{v,N,K}$ is a tight frame in $ell^2(\mathbf{Z})$ than any f in the space can be represented as

$$f = \frac{N}{K}\sum_{l,m}\langle f, v_{lm}\rangle v_{lm}.$$

The requirement that $\Phi_{v,N,K}$ is a tight frame imposes a number of nonlinear constraints on the window v. Designing the window v then requires optimization of its characteristics (time-frequency localization) under these constraints. In this paper we propose a procedure which facilitates this design even for very long filters by taking advantage of redundancy. The procedure also gives long filters which can be specified with a few parameters.

[1] $\mathbf{Z}_K$ here denotes the set $\{0, 1, ..., K\}$.

2. WINDOW DESIGN FOR TIGHT WEYL-HEISENBERG FRAMES: PROBLEM FORMULATION

A family of vectors, $\{\varphi_j\}_{j\in J}$ in a Hilbert space is said to be a tight frame if for any f in the space $\sum_{j\in J} |< f, \varphi_j >|^2 = A\|f\|^2$, for some constant $A > 0$. If $\{\varphi_j\}_{j\in J}$ is a tight frame, any f in the space can be represented as

$$f = \frac{1}{A}\sum_{j\in J}\langle f, \varphi_j\rangle\varphi_j. \tag{4}$$

A tight frame with the frame constant $A = 1$ is an orthonormal basis (we assume that the frame vectors are normalized to unit norm). In general, for redundant frames the frame constant A is greater than 1, and it represents redundancy of the frame. For a tight Weyl-Heisenberg frame, $\Phi_{v,N,K}$, its redundancy is equal to the ratio K/N.

A Weyl-Heisenberg family, $\Phi_{v,N,K}$, forms a tight frame in $\ell^2(\mathbf{Z})$ if and only if the window satisfies the following constraints [9, 3]

$$\sum_{j\in\mathbf{Z}} v[n+jN]v[n+jN+iK] = \frac{1}{N}\delta[i],\ n = 0, 1, ..., N-1. \tag{5}$$

The issue in designing a window for short-time Fourier analysis is to attain a high concentration of its energy around the origin in the time-frequency plane. In many engineering applications energy leakage of a filter out of a prescribed frequency band is a more relevant design criterion than is a time-bandwidth product. Accordingly, design of the window v will be here directed towards maximizing its energy in the $[0, \frac{\pi}{K}]$ frequency band, given its length. For a filter v of length L, its energy in the band $[0, \frac{\pi}{K}]$ is given by

$$E = \mathbf{v}^T\mathbf{S}_{(K,L)}\mathbf{v}, \tag{6}$$

where $\mathbf{v}$ is the column vector

$$\mathbf{v} = [v(0)v(1)...v(L-1)]^T$$

and $\mathbf{S}_{(K,L)}$ is the $L\times L$ matrix

$$[\mathbf{S}_{(K,L)}]_{i,j} = \frac{1}{\pi}\frac{sin(\alpha(i-j))}{(i-j)}$$

with $\alpha = \pi/K$. So the design amounts to maximizing the energy function given by the quadratic form in (6) under constraints given in (5).

A straightforward approach would be to use a constrained optimization procedure. This requires maximization of the quadratic form of L variables under LN/K quadratic constraints. Specification of the designed windows requires identifying all L window taps.

Alternatively, if closed form solutions for the constraints in (5) are known, an unconstrained optimization procedure can be used. A complete set of solutions of the tight frame constraints can be given through a parameterization of paraunitary matrices, based for instance on Given's rotations [4, 3]. In order to be able to express the numerical complexity of the parametric approach concisely, assume that K is a multiple of N. In that case, window design requires unconstrained optimization of the energy function over the space of $L - LN/K$ rotation angles, and as many parameters are needed to specify the obtained windows. Note, that the energy function is a very complex trigonometric function of Given's rotation angles.

Here we propose a design method which amounts to solving a system of LN/K quadratic equations in LN/K unknowns that fully describe designed windows. The proposed approach has clear advantages in cases with relatively high redundancy factors, K/N, in the sense that the numerical complexity of the design algorithm is significantly reduced and that resulting windows can be concisely described with only a few parameters. For example, in the extreme case, $N = 1$, of frames $\Phi_{v,1,256}$ with windows of length $L = 1024$, the constrained optimization procedure requires optimization in the space of 1024 filter taps under 4 quadratic constraints, the parametric approach requires unconstrained optimization of the energy function over the space of 1020 rotation angles, whereas the method proposed in the next section amounts to solving a system of 4 quadratic equations in 4 unknowns. Furthermore describing the designed windows requires 1024, 1020 or 4 parameters, respectively.

3. NEW DESIGN PROCEDURE

The idea behind the proposed method is to represent the window v as the linear combination of eigenvectors ρ_i of the matrix $\mathbf{S}_{(K,L)}$,

$$v = \sum_{i=0}^{L-1}\alpha_i\rho_i. \tag{7}$$

Eigen structures of matrices $\mathbf{S}_{(K,L)}$ were studied by Slepian [10]. The eigenvectors $\rho_0, \rho_1, ..., \rho_{L-1}$ are obtained by truncating in time certain *prolate spheroidal sequences* [10]. They form an orthonormal basis of $\mathbf{R}^L$, so any window of length L can be represented as their linear combination. Corresponding eigenvalues are distinct, real and positive and we order them so that

$$\lambda_0 > \lambda_1 > ... > \lambda_{L-1} > 0.$$

The windows that we are interested in are those which are well concentrated in low frequencies, and these are basically linear combinations of eigenvectors ρ_i which are themselves well concentrated. A measure of the concentration of ρ_i in frequency is given by the corresponding eigenvalue. Namely the total energy of ρ_i in the $[0, \frac{\pi}{K}]$ frequency band is equal to λ_i (note that the eigenvectors are normalized to unit norm). It turns out that L/K eigenvectors of $\mathbf{S}_{(K,L)}$ have most of their energy in the $[0, \frac{\pi}{K}]$ band [10], i.e. the first L/K eigenvalues are greater than 0.5 and $L/K - 1$ of them are close to 1. The rest of the eigenvalues, λ_i, decrease rapidly towards zero as i increases beyond L/K.

The design requires solving for a set of expansion coefficients α_i in (7) so that the tight frame constraints in (5) are satisfied. As there are LN/K constraints, the window v has to be represented as a linear combination of at least LN/K eigenvectors ρ_i, and we take $\rho_0, \rho_1, ..., \rho_{LN/K-1}$ since they have the best localization in low frequencies. The design constraints translate into the following system of LN/K quadratic equations in the expansion coefficients,

$$\sum_{l,m=0}^{KL/N-1} c_{lm}^{(ik)}\alpha_l\alpha_m = \frac{1}{N}\delta[k],\ 0\le i < N, 0 \le k < \frac{L}{K}, \tag{8}$$

where

$$c_{lm}^{(ik)} = \sum_j \rho_l[i+jN]\rho_m[i+jN+kK].$$

The design procedure then requires finding solutions of this system corresponding to windows that give high values of the optimization criterion in (6). This is particularly easy when LN/K is small, since the system, up to sign factor, has at most $2^{(LN/K)-1}$ different solutions. For systems of higher orders, good solutions are usually obtained if iterative procedures for solving this system are started with the initial values, $[\alpha_0^0\ \alpha_1^0\ \ldots\ \alpha_{\frac{LN}{K}-1}]$, which have all of their "energy" concentrated in coefficients with low indices.

A nice property of vectors ρ_i is that even indexed vectors are symmetric. So if v is required to be symmetric then it can be represented as the linear combination of the first LN/K even indexed vectors, $\rho_0, \rho_2, \ldots, \rho_{2(LN/K)-2}$.

In order order to the satisfy design constraints when LN/K is considerably larger than L/K, v needs to be represented using many vectors ρ_i which have 0.00% energy in the band, and that can have a bad impact on its frequency localization. However, as the set of windows which are well localized in frequency is practically spanned by the first L/K eigenvectors, all good windows are close to the linear span of these vectors. Therefore, there are always solutions where coefficients corresponding to the eigenvectors with high indices have insignificant values, and these solutions give good windows.

4. DESIGN EXAMPLES

The price paid by this new design procedure is that it does not attain the global maximum of the design criterion. However, this loss in energy concentration of the window at low frequencies is not significant. In Figure 1, for comparison we plot curves which represent the amount of energy of tight frame windows in the frequency band $[0, \pi/K]$ for window lengths $L = nK,\ n = 1, 2, \ldots, 7$, when the subsampling factor $N = 1$, i.e. the most redundant case for a given K. The upper curve corresponds to windows with the highest energy content in the band, obtained using the constrained optimization procedure. The lower curve represents windows obtained using the new method, and we can see that the difference is insignificant. As the subsampling factor N increases, the difference stays within few percent. These curves do not depend on K (inverse of the frequency resolution of $\Phi_{v,N,K}$) but only on the ratio LN/K. So, the curve for the constrained optimization procedure is obtained for $K = 16$, since complexity of this procedure increases with K and it becomes hardly implementable for large values of K.

The new design procedure has pertained so far to windows which are represented using minimal number, LN/K, of expansion vectors needed to satisfy design constraints. Design results can be improved if additional vectors are allowed, and windows are optimized in the space of $LN/K + k$ expansion vectors ρ_i. Since vectors the ρ_i form an orthonormal basis for $\mathbf{R}^L$, in this manner we can approach the optimal windows arbitrarily closely. For example, for the case $\Phi_{v,1,256}$, the best localized window of length 1024, represented with 4 (minimal number) expansion vectors has 92.56% of its energy in the $[0, \pi/K]$ band. With 6 expansion vectors we attain a window with 93.15% of its energy in the band. Note that specifying these two windows requires 4 and 6 coefficients respectively, while for windows obtained from the constrained optimization, specification requires all 1024 filter taps. The best filters for frames $\Phi_{v,1,K}$ of length $L = 4K$ obtained using the constrained optimization procedure have 93.49% of their energy in the band (this is the result obtained for $K = 8, 16, 32$). So, the new procedure gives slightly suboptimal design results with low complexity, but also leaves space for improvements based on the trade off between complexity and quality of design.

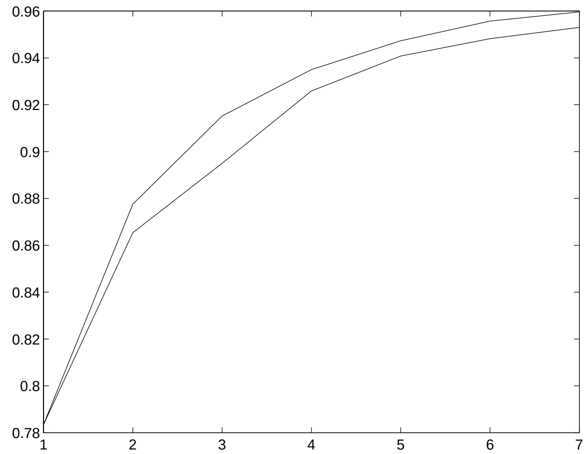

Figure 1: *Concentration of windows of tight frames $\Phi_{v,1,K}$ in the frequency band $[0, \pi/K]$, for filters of length $L = nK,\ n = 1, 2, \ldots, 7$. Top curve - windows obtained from the constrained optimization procedure. Bottom curve - windows obtained using the design method based on prolate spheroidal sequences.*

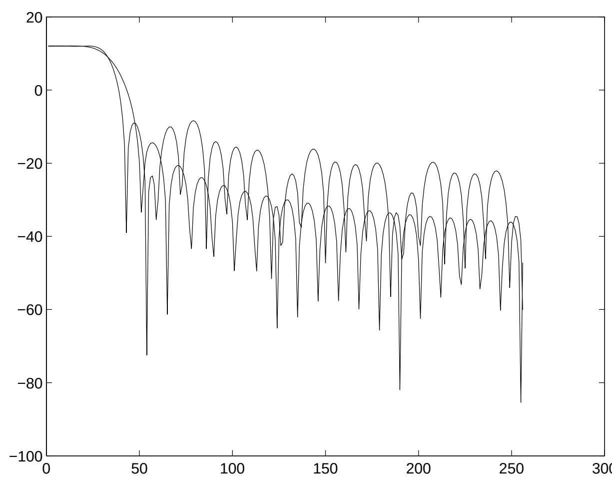

Figure 2: *Magnitude responses (log-plots) of windows for frames $\Phi_{v,1,K}$ of length $L = 6K$. The window with sharper cut-off is obtained with the constrained optimization procedure, whereas the window with higher attenuation in the stop band is obtained using the new design method.*

Filters obtained from the new design procedure, with the minimal number of parameters, usually have larger bandwidth than filters obtained with the direct constrained optimization, however they often have higher attenuation in the stop-band. This is illustrated in Figure 2, for symmetric filters for frames $\Phi_{v,1,K}$ of length $L = 6K$.

In Figure 3 we show the time domain plot of a symmetric window for a tight frame $\Phi_{v,64,256}$. The length of this window is $L = 512$ ($L = 2K$). This window has poor frequency localization, i.e. only 83.31% of its energy is in the $[0, \pi/256]$ band.

Note that this is because for better localization longer filters are needed, and that using the constrained optimization procedure for symmetric filters of length $L = 2K$ for frames $\Phi_{v,1,K}$, we were not able to find filters with more than 83.64% of energy in the band. The reason we show this filter is to give an example of the design when the number of expansion vectors needed to satisfy the design constraints is significantly smaller than the number of constraints. In this case, the total number of design constraints is 128, and the shown filter is represented using only first 6 vectors ρ_i while satisfying the constraints with accuracy of order 10^{-12}.

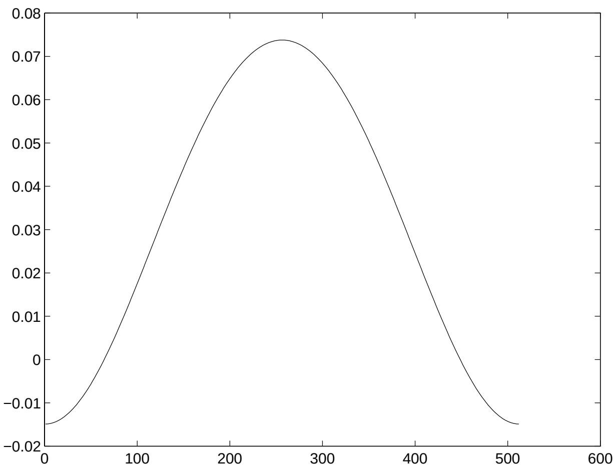

Figure 3: *A window of length 512 for a tight frame $\Phi_{v,64,256}$. The total number of constraints in this case is 128, and this window satisfies them accurately while being represented with only 6 expansion coefficients.*

The magnitude response of a window for a tight frame $\Phi_{v,16,256}$ is plotted in Figure 4. The filter length is $L = 1024$. The tight frame constraints are satisfied with precision 10^{-9}. For comparison, on the same graph we plot the magnitude response of the rectangular window of length $L = 256$, which is the only finite length window which makes $\Phi_{v,256,256}$ an orthonormal basis [3]. This demonstrates the advantages of introducing redundancy in both allowing for windows with better frequency selectivity and for facilitating design of long filters using the proposed algorithm.

5. CONCLUSION

In this paper, we proposed a method for designing windows for discrete-time Weyl-Heisenberg frames. This method has general applicability to the design of low-pass filters under nonlinear constraints. It is particularly appealing when the total number of constraints is not large, as in that case it allows for computationally inexpensive designs of long filters and specifications of the designed filters with only a few parameters.

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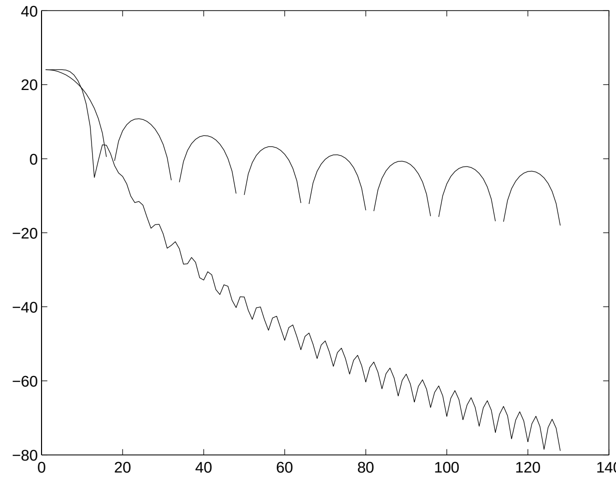

Figure 4: *Magnitude response (log plot) of a 1024 long filter for a tight frame $\Phi_{v,16,256}$. For comparison, the magnitude response of the only FIR window giving an orthonormal basis $\Phi_{v,256,256}$ is also plotted (the less selective filter).*

6. REFERENCES

[1] J. B. Allen, "Short Term Spectral Analysis, Synthesis and Modification by Discrete Fourier Transform", *IEEE Trans. Signal Processing*, Vol. ASSP-25, No. 3, pp.235-238, June 1977.

[2] R. Balian,"Un principe d'incertitude fort en théorie du signal on mécanique quantique", *C. R. Acad. Sc. Paris*, vol.292, série 2, 1981.

[3] Z. Cvetković,"On Discrete Short-Time Fourier Analysis", *preprint*, September 1997.

[4] Z. Cvetković and M. Vetterli, "Tight Weyl-Heisenberg Frames in $\ell^2(\mathbf{Z})$", *IEEE Trans. Signal Processing*, to appear.

[5] I. Daubechies,*Ten Lectures on Wavelets*, CBMS-NSF Series in Appl. Math, SIAM, 1992.

[6] R. J. Duffin and A. C. Schaeffer,"A Class of Nonharmonic Fourier Series", *Trans. Amer. Math. Soc.*, Vol.72, pp.341-366, March 1952.

[7] D. Gabor, "Theory of Communications", *J. IEE*, Vol.93 (III), pp.429-457, 1946.

[8] F. Low, "Complete Sets of Wave Packets", in *A Passion for Physics - Essays in Honor of Geoffrey Chew*, pp.17-22, Singapore: World Scientific, 1985.

[9] M. R. Portnoff,"Time-Frequency Representations of Digital Signals and Systems Based on Short-Time Fourier Analysis",*IEEE Trans. Acoustics Speech and Signal Processing*,Vol.28, No.1, pp.55-69, February 1980.

[10] D. Slepian,"Prolate Spheroidal Wave Functions, Fourier Analyis and Uncertainty V:the Discrete Case", *Bell Systems Technical Journal*, Vol.57, 1371-1430, 1978.

[11] M. Vetterli,"Filter Banks Allowing Perfect Reconstruction", *Signal Processing*,Vol.10, No.3, pp.219-244, April 1986.