NEW HIGHER ORDER SPECTRA AND TIME-FREQUENCY REPRESENTATIONS FOR DISPERSIVE SIGNAL ANALYSIS*

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ABSTRACT

For analysis of signals with arbitrary dispersive phase laws, we extend the concept of higher order moment functions and define their associated higher order spectra. We propose a *new* higher order time-frequency representation (TFR), the higher order generalized warped Wigner distribution (HOG-WD). The HOG-WD is obtained by warping the previously proposed higher order Wigner distribution, and is important for analyzing signals with arbitrary time-dependent instantaneous frequency. We discuss links to prior higher order techniques and investigate properties of the HOG-WD. We extend the HOG-WD to a class of higher order, alternating sign, frequency-shift covariant TFRs. Finally, we demonstrate the advantage of using the generalized higher order spectra to detect phase coupled signals with dispersive instantaneous frequency characteristics.

1. INTRODUCTION

Higher order (HO) moment/cumulant functions and their multidimensional Fourier transform pairs, HO spectra, of a random process $\tilde{x}(t)$ [7, 6] are the Nth order extensions of the second order concepts of the autocorrelation function, and its Fourier transform pair, the power spectrum. In [9], the HO moment functions and HO spectra were warped hyperbolically [8] to obtain scale HO moment functions and their multidimensional Mellin transform pairs, scale HO spectra, useful for the analysis of signals with hyperbolic instantaneous frequency (IF) laws such as Doppler invariant signals.

The concept of using an *arbitrary* warping to map one signal geometry onto a new geometry has been used in the context of time-frequency analysis [2, 8]. In this paper, we apply the warping approach, based on a warping function $\xi(c)$, to obtain *new* HO generalized (HOG) moment functions and their multidimensional generalized Fourier transform, HOG spectra. These new HO statistics are matched to signals with arbitrary dispersive IF laws, that depend on $\xi(c)$. Special cases of the HOG spectrum for order $N = 3$ include the classical bispectrum in [7], the scale bispectrum in [9], and the *new* power warped bispectrum and *new* exponential bispectrum developed in this paper. Also, we provide an example demonstrating the advantage of the HOG spectrum for detecting phase coupled sinusoids with dispersive IF characteristics.

The HO-Wigner distribution (HO-WD) in [4, 10, 5] can be viewed as a time-varying counterpart of HO spectra, and is important for analyzing signals whose phase law varies linearly with time. The Nth order HO-WD [4], of a deterministic signal $x(t)$ is a time/multi-frequency representation that is defined as

$$\text{HO-WD}_x^N(t, \underline{\boldsymbol{f}}) = \int_{\tau_1}\!\cdots\!\int_{\tau_{N-1}} u_x^N(t, \underline{\boldsymbol{\tau}}) \prod_{i=1}^{N-1} e^{-j2\pi f_i \tau_i} d\tau_i \qquad (1)$$

the $(N-1)$th dimensional Fourier transform[1] of a local Nth order moment function[2,3] $u_x^N(t, \underline{\boldsymbol{\tau}}) = x^*(t-\alpha) \prod_{n=1}^{N-1} (\mathcal{L}_n x)(t-\alpha+\tau_n)$, with centering parameter $\alpha = \frac{1}{N}\sum_{l=1}^{N-1} \tau_l$. The conjugation operator $\mathcal{L}_n$ conjugates the signal if the index n is even, e.g. $(\mathcal{L}_4 x)(t) = x^*(t)$. The HO-WD in (1) preserves constant time shifts, and under certain conditions, constant frequency shifts on the signal. The third order HO-WD [5] has been used to detect transient signals in [4]. An entire class of HO-time frequency representations (TFRs) was derived in [4, 10] by smoothing the HO-WD with a multidimensional kernel; it can be viewed as the Nth order extension of Cohen's class of quadratic TFRs [3].

In [9], the HO-Altes Q-distribution (HO-QD) was introduced as an extension of the HO-WD to the scale domain. Alternatively, it is an extension of the second order Altes Q-distribution [1] to higher order. The HO-QD preserves scale changes and under certain conditions, hyperbolic frequency shifts. It is important for analyzing signals with hyperbolic IF characteristics, such as bat echolocation pulses.

In [8], it was shown that quadratic TFRs which are matched to signals with dispersive IF characteristics can be obtained by warping the well-known quadratic Wigner distribution (WD), using a warping function, $\xi(c)$. The resulting generalized warped WD (GWWD) [8] is a quadratic TFR based on second order moments of the signal. The GWWD can be viewed as a time-varying generalized power spectrum. Special cases of the GWWD include the WD for $\xi(c) = c$ and the Altes Q-distribution [1] for $\xi(c) = \ln c$.

In this paper, we unify and generalize the concepts of HO-TFRs using warping techniques. Equivalently, we extend to higher order the quadratic GWWD. The *new* higher order generalized warped Wigner distribution (HOG-WD) is well-suited for analyzing signals with dispersive IF characteristics. Important special cases of the HOG-WD include the HO-WD developed in [4, 10] (matched to signals with constant IF), the HO-QD developed in [9] (matched to signals with hyperbolic IF), the *new* HO power warped WD (matched to signals with power law IF) proposed in this paper, and the *new* HO exponentially warped WD (matched

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[1] Unless otherwise specified, integration limits are -∞ to ∞.

[2] This local higher order moment is similar to the one in [4] except we conjugate every other signal term in order to produce frequency shift covariance.

[3] For space considerations, vector functions are used in the arguments of multidimensional functions. Vectors are denoted using boldfaced and underlined text, e.g. $\underline{\boldsymbol{\tau}} = [\tau_1, \cdots, \tau_{N-1}]$.

to signals with exponential IF) also proposed in this paper. In this paper, we define the *new* HOG-WD and we list some of its desirable properties and special signal cases. Furthermore, we extend the HOG-WD to a broader class of HO-TFRs that for even N preserve dispersive, alternating sign frequency shifts.

2. HIGHER ORDER GENERALIZED SPECTRA FOR RANDOM SIGNALS

2.1. Definition of Higher Order Generalized Spectra

A random signal $\tilde{x}(t)$ is Nth order wide sense stationary if its Nth order moment function, $r_{\tilde{x}}^N(\underline{\boldsymbol{\tau}}) = \mathbf{E}\,[v_{\tilde{x}}^N(t,\underline{\boldsymbol{\tau}})]$, depends only on the lags $\tau_i, i = 1,\cdots,N$. Here, $v_{\tilde{x}}^N(t,\underline{\boldsymbol{\tau}}) = \tilde{x}^*(t)\prod_{n=1}^{N-1}\tilde{x}(t+\tau_n)$ is the local moment function [7] and $\mathbf{E}\,[\cdot]$ is the expectation operator. The Nth order moment spectrum is defined as the $(N-1)$th dimensional Fourier transform of $r_{\tilde{x}}^N(\underline{\boldsymbol{\tau}})$, i.e.

$$R_{\tilde{x}}^N(\underline{\boldsymbol{f}}) = \int_{\tau_1}\!\cdots\!\int_{\tau_{N-1}} r_{\tilde{x}}^N(\underline{\boldsymbol{\tau}}) \prod_{i=1}^{N-1} e^{-j2\pi f_i \tau_i} d\tau_i. \qquad (2)$$

In [2, 8], arbitrary warpings were used to map TFRs matched to one signal geometry onto new TFRs matched to a new signal geometry. In this paper, we apply the warping idea, using a warping function $\xi(c)$, to generalize the HO moment functions and HO spectra. These new HOG statistics are matched to signals with dispersive IF laws dependent on $\xi(c)$. We obtain the HOG moment function as

$$\begin{aligned} s_{\tilde{x}}^{N,\xi}(\underline{\boldsymbol{\sigma}}) &= \mathbf{E}\,[v_{\mathcal{W}_\xi \tilde{x}}^N(t_r\xi(\tfrac{t}{t_r}), t_r\xi(\sigma_1),\cdots,t_r\xi(\sigma_{N-1}))] \\ &= \mathbf{E}\,[\tilde{x}^*(t)\prod_{n=1}^{N-1}\tilde{x}\Big(t_r\xi^{-1}\Big(\xi(\tfrac{t}{t_r})+\xi(\sigma_n)\Big)\Big)] \qquad (3) \end{aligned}$$

where $(\mathcal{W}_\xi\tilde{x})(t) = \tilde{x}(t_r\xi^{-1}(\frac{t}{t_r}))$ is the signal warping operator, $t_r > 0$ is a fixed reference time, and $\xi(c) \in \mathbb{R}$ is a one-to-one, differentiable function such that $\xi^{-1}(\xi(c)) = c$. Note that if the Nth HOG moment function, $s_{\tilde{x}}^{N,\xi}(\underline{\boldsymbol{\sigma}})$, depends only on the lags $\sigma_i, i = 1,\cdots,N-1$, then we say it is Nth order wide sense ξ-stationary. The HOG spectrum $S_{\tilde{x}}^{N,\xi}(\underline{\boldsymbol{\gamma}})$ is defined as the $(N-1)$th dimensional generalized Fourier transform of $s_{\tilde{x}}^{N,\xi}(\underline{\boldsymbol{\sigma}})$ in (3):

$$\begin{aligned} S_{\tilde{x}}^{N,\xi}(\underline{\boldsymbol{\gamma}}) &= \mathcal{P}_{\sigma_1\to\gamma_1}\cdots\mathcal{P}_{\sigma_{N-1}\to\gamma_1}\{s_{\tilde{x}}^{N,\xi}(\underline{\boldsymbol{\sigma}})\} \\ &= \int_{\sigma_1}\!\cdots\!\int_{\sigma_{N-1}} s_{\tilde{x}}^{N,\xi}(\underline{\boldsymbol{\sigma}})\prod_{i=1}^{N-1}e^{-j2\pi\gamma_i\xi(\sigma_i)}\,|\xi'(\sigma_i)|\,d\sigma_i. \; (4) \end{aligned}$$

The 1-D generalized transform of $s_{\tilde{x}}(\sigma)$ is defined as:

$$\mathcal{P}_{\sigma\to\gamma}\{s_{\tilde{x}}(\sigma)\} = \rho_{s_{\tilde{x}}}(\gamma) \triangleq \int s_{\tilde{x}}(\sigma)\,e^{-j2\pi\gamma\xi(\sigma)}\;|\xi'(\sigma)|\,d\sigma \qquad (5)$$

where $\xi'(c) = \frac{d}{dc}\xi(c)$. For example, $\mathcal{P}$ is the familiar Fourier transform when $\xi(c) = c$, whereas $\mathcal{P}$ is the Mellin transform when $\xi(c) = \ln c$. In Table 1, we list some special cases of the HOG moment functions in (3) and their associated HOG spectra in (4) for order $N = 3$ and for various $\xi(c)$. For example, in row 4 we obtain the *new* HO power warped spectra for $\xi(c) = \xi_\kappa(c) = \mathrm{sgn}(c)\;|c|^\kappa$, where $\mathrm{sgn}(c)$ is the signum function.

Alternatively, given the conventional HO spectrum, $R_{\tilde{x}}^N(\underline{\boldsymbol{f}})$ in (2), we can derive the HOG spectrum by (i) warping the signal to obtain $(\mathcal{W}_\xi\tilde{x})(t)$, (ii) applying the HO spectrum to the warped signal, $R_{\mathcal{W}_\xi\tilde{x}}^N(\underline{\boldsymbol{f}})$, and (iii) scaling the axes to obtain the correct units, i.e.

$$S_{\tilde{x}}^{N,\xi}(\underline{\boldsymbol{\gamma}}) \triangleq \frac{1}{t_r^{N-1}}R_{\mathcal{W}_\xi\tilde{x}}^N(\frac{1}{t_r}\;\underline{\boldsymbol{\gamma}}) \triangleq \frac{1}{t_r^{N-1}}R_{\mathcal{W}_\xi\tilde{x}}^N(\frac{\gamma_1}{t_r},\cdots,\frac{\gamma_{N-1}}{t_r})$$

This is a useful formulation for implementing any HOG spectra as we can use the standard algorithms for HO spectra. HOG spectra are important in analyzing phase coupled sinusoids with arbitrary IF as we demonstrate next.

2.2. Illustrative Example

We show that phase coupled, complex sinusoids with κth power law IF can be characterized by the $N = 3$, κth *power* warped moment function and the κth *power* warped bispectrum. Define $\tilde{x}(t) = e^{j(2\pi b_1\xi_\kappa(\frac{t}{t_r})+\Theta_1)} + e^{j(2\pi b_2\xi_\kappa(\frac{t}{t_r})+\Theta_2)}$ where $\xi_\kappa(c) = \mathrm{sgn}(c)\;|c|^\kappa$, $b_1, b_2 \in \mathbb{R}$, $b_2 = 2b_1$, and the phases Θ_1, Θ_2 are uniformly distributed on $[-\pi,\pi]$. If $\tilde{x}(t)$ is phase coupled, then $\Theta_2 = 2\Theta_1$; otherwise, Θ_1 is independent of Θ_2. For both the phase coupled and non-phase coupled cases, the $N = 2$, κth power warped moment functions in (3) are identical and given by: $s_{\tilde{x}}^{2,\xi_\kappa}(\sigma) = \mathbf{E}\,[\tilde{x}^*(t)\tilde{x}(t_r\xi_{\frac{1}{\kappa}}(\xi_\kappa(\frac{t}{t_r})+\xi_\kappa(\sigma))] = e^{j2\pi b_1\xi_\kappa(\frac{t}{t_r})} + e^{j2\pi 2b_1\xi_\kappa(\frac{t}{t_r})}$. Hence, the $N = 2$, κth power warped spectra in (4), $S_{\tilde{x}}^{2,\xi_\kappa}(\gamma) = \delta(\gamma-b_1)+\delta(\gamma-2b_1)$, are also identical and cannot be used to detect phase coupled power sinusoids. The $N = 3$, κth power warped moment function, $s_{\tilde{x}}^{3,\xi_\kappa}(\sigma_1,\sigma_2)$, in row 4 of Table 1, however, can be used to detect phase coupled power sinusoids because it is only zero in the non-phase coupled case:

$$s_{\tilde{x}}^{3,\xi_\kappa}(\sigma_1,\sigma_2) = \begin{cases} 0, & \Theta_1 \text{ independent } \Theta_2 \\ e^{j2\pi b_1(\xi_\kappa(\sigma_1)+\xi_\kappa(\sigma_2))}, & \Theta_2 = 2\Theta_1. \end{cases}$$

For the phase coupled case, the κth power warped bispectrum, in row 4 of Table 1, simplifies to a 2-D impulse located at (b_1, b_1):

$$S_{\tilde{x}}^{3,\xi_\kappa}(\gamma_1,\gamma_2) = \begin{cases} 0, & \Theta_1 \text{ independent } \Theta_2 \\ \delta(\gamma_1-b_1,\gamma_2-b_1), & \Theta_2 = 2\Theta_1. \end{cases}$$

This is important for detection applications since it will indicate one peak for only the case of the phase coupled κth-power sinusoids. Since this holds for all κ, it includes the special cases of sinusoids ($\kappa = 1$) and linear FM chirps ($\kappa = 2$).[4]

3. HIGHER ORDER GENERALIZED WARPED WIGNER DISTRIBUTION (HOG-WD)

3.1. Definition of the HOG-WD

We also apply the warping concept, based on the warping function $\xi(c)$, to HO-TFRs. We obtain the *new* Nth order HOG-WD by: (i) warping the deterministic signal $x(t)$, (ii) computing the HO-WD in (1) of the warped signal, $(\mathcal{W}_\xi x)(t)$, and (iii) transforming the time/multi-frequency axes for correct time/multi-frequency localization. Thus,

$$\text{HOG-WD}_x^{N,\xi}(t,\underline{\boldsymbol{f}}) = \text{HO-WD}_{\mathcal{W}_\xi x}^{N,}\Big(t_r\xi(\frac{t}{t_r}),\frac{1}{t_r\nu(t)}\;\underline{\boldsymbol{f}}\Big) \qquad (6)$$

where $\nu(t) = \frac{d}{dt}\xi(\frac{t}{t_r})$ and $(\mathcal{W}_\xi x)(t)$ is defined in (3). Expanding (6), we can re-express the HOG-WD as

$$\begin{aligned} \text{HOG-WD}_x^{N,\xi}(t,\underline{\boldsymbol{f}}) &= \mathcal{P}_{\sigma_1\to\frac{f_1}{\nu(t)}}\cdots\mathcal{P}_{\sigma_{N-1}\to\frac{f_{N-1}}{\nu(t)}}\{M_x^{N,\xi}(t,\underline{\boldsymbol{\sigma}})\} \\ &= t_r^{N-1}\int_{\sigma_1}\!\cdots\!\int_{\sigma_{N-1}}M_x^{N,\xi}(t,\underline{\boldsymbol{\sigma}})\prod_{n=1}^{N-1}e^{-j2\pi\frac{f_n}{\nu(t)}\xi(\sigma_n)}|\xi'(\sigma_n)|\;d\sigma_n, \; (7) \end{aligned}$$

an $(N-1)$th dimensional transform of the warped local moment function $M_x^{N,\xi}(t,\underline{\boldsymbol{\sigma}})$, and $\underline{\boldsymbol{\sigma}} = [\sigma_1,\cdots,\sigma_{N-1}]$. The integration limits depend on the range of $\xi(c)$. The warped local moment function, $M_x^{N,\xi}(t,\underline{\boldsymbol{\sigma}})$ in (7), is a warped version of $u_x^N(t,\underline{\boldsymbol{\tau}})$ in (1):

[4]For this zero mean signal example, either power warped moment functions or power warped cumulants could have been used as they are identical for $N \leq 3$. For random signals, cumulant functions are sometimes preferred as their spectra satisfies such desirable properties as suppressing Gaussian noise and characterizing a signal's phase information [7].

$\xi(c)$	3rd order Generalized moment function $s_{\tilde{x}}^{3,\xi}(\sigma_1,\sigma_2)$	Generalized bispectra $S_{\tilde{x}}^{3,\xi}(\gamma_1,\gamma_2)$	ref.
$\xi_{\text{linear}}(c)=c$	$\mathbf{E}\left[\tilde{x}^*(t)\tilde{x}(t+t_r\sigma_1)\tilde{x}(t+t_r\sigma_2)\right]$	$\int\int s_{\tilde{x}}^{3,\xi_{\text{linear}}}(\sigma_1,\sigma_2)e^{-j2\pi(\gamma_1\sigma_1+\gamma_2\sigma_2)}d\sigma_1 d\sigma_2$	[6, 7]
$\xi_{\ln}(c)=\ln(c)$	$\mathbf{E}\left[\tilde{x}^*(t)\tilde{x}(t\sigma_1)\tilde{x}(t\sigma_2)\right]$	$\int_0^\infty\int_0^\infty s_{\tilde{x}}^{3,\xi_{\ln}}(\sigma_1,\sigma_2)e^{-j2\pi(\gamma_1\ln(\sigma_1)+\gamma_2\ln(\sigma_2))}\frac{d\sigma_1}{\sigma_1}\frac{d\sigma_2}{\sigma_2}$	[9]
$\xi_\kappa(c)=\text{sgn}(c)\ \lvert c\rvert^\kappa$	$\mathbf{E}\left[\tilde{x}^*(t)\tilde{x}(t_r\xi_{\frac{1}{\kappa}}(\xi_\kappa(\frac{t}{t_r})+\xi_\kappa(\sigma_1)))\tilde{x}(t_r\xi_{\frac{1}{\kappa}}(\xi_\kappa(\frac{t}{t_r})+\xi_\kappa(\sigma_2)))\right]$	$\int\int s_{\tilde{x}}^{3,\xi_\kappa}(\sigma_1,\sigma_2)e^{-j2\pi(\gamma_1\xi_\kappa(\sigma_1)+\gamma_2\xi_\kappa(\sigma_2))}\kappa^2\lvert\sigma_1\sigma_2\rvert^{\kappa-1}d\sigma_1 d\sigma_2$	new
$\xi_{\exp}(c)=e^c$	$\mathbf{E}\left[\tilde{x}^*(t)\tilde{x}(t_r\ln(e^{(\frac{t}{t_r})}+e^{\sigma_1}))\tilde{x}(t_r\ln(e^{(\frac{t}{t_r})}+e^{\sigma_2}))\right]$	$\int\int s_{\tilde{x}}^{3,\xi_{\exp}}(\sigma_1,\sigma_2)e^{-j2\pi(\gamma_1 e^{\sigma_1}+\gamma_2 e^{\sigma_2})}e^{(\sigma_1+\sigma_2)}d\sigma_1 d\sigma_2$	new
one-to-one	$\mathbf{E}\left[\tilde{x}^*(t)\tilde{x}(t_r\xi^{-1}(\xi(\frac{t}{t_r})+\xi(\sigma_1))\tilde{x}(t_r\xi^{-1}(\xi(\frac{t}{t_r})+\xi(\sigma_2))\right]$	$\int\int s_{\tilde{x}}^{3,\xi}(\sigma_1,\sigma_2)e^{-j2\pi(\gamma_1\xi(\sigma_1)+\gamma_2\xi(\sigma_2))}\ \lvert\xi'(\sigma_1)\xi'(\sigma_2)\rvert\ d\sigma_1 d\sigma_2$	new

Table 1: *Third order HOG moment functions in (3) and their corresponding HOG bispectra in (4) for different $\xi(c)$. Here, $\mathbf{E}[\cdot]$ is the expectation operator. Rows 2–5 are special cases of the last row.*

$$M_x^{N,\xi}(t,\underline{\boldsymbol{\sigma}}) = u^N_{\mathcal{W}_\xi x}\Big(t_r\xi(\frac{t}{t_r}),t_r\xi(\sigma_1),\cdots,t_r\xi(\sigma_{N-1})\Big)$$
$$= x^*\Big(t_r\xi^{-1}\Big(\xi(\frac{t}{t_r})-\frac{1}{N}\sum_{i=1}^{N-1}\xi(\sigma_i)\Big)\Big)$$
$$\prod_{n=1}^{N-1}(\mathcal{L}_n x)\Big(t_r\xi^{-1}\Big(\xi(\frac{t}{t_r})+\xi(\sigma_n)-\frac{1}{N}\sum_{i=1}^{N-1}\xi(\sigma_i)\Big)\Big). \qquad (8)$$

Alternatively, we can express the HOG-WD in (7) in terms of the generalized transform, $\rho_x(c)$ in (5):

$$\text{HOG-WD}_x^{N,\xi}(t,\underline{\boldsymbol{f}}) = \frac{1}{t_r}\int_c \rho_x^*\Big(\Big[\frac{1}{\nu(t)}\sum_{i=1}^{N-1}f_i\Big]-\frac{c}{N}\Big)$$
$$\prod_{n=1}^{N-1}(\mathcal{L}_n\rho_x)\Big(\Big[\frac{f_n}{\nu(t)}+\frac{c}{N}\Big](-1)^{n+1}\Big)e^{j2\pi c\xi(\frac{t}{t_r})}dc. \qquad (9)$$

3.2. Importance of the HOG-WD

The HOG-WD is an important HO-TFR that could be used for the multidimensional analysis of signals with dispersive IF characteristics. In particular, let $x_b(t)=e^{j2\pi b\xi(t/tr)}$, $b\in\mathbb{R}$, be a complex exponential with arbitrary IF, $b\,\nu(t)=b\,\frac{d}{dt}\xi(\frac{t}{t_r})$. Then, for even order N, the HOG-WD of this signal is ideally centered at $\pm b\nu(t)$, along its multi-frequency axes, i.e.

$\text{HOG-WD}_{x_b}^{N,\xi}(t,\underline{\boldsymbol{f}}) = \lvert t_r\nu(t)\rvert^{1-N}\prod_{n=1}^{N-1}\delta(f_n+(-1)^n b\nu(t))$,

where $\delta(t)$ is the Dirac delta function. Notice the alternating sign in the argument of the delta function. For example, if the warping function $\xi(c)=\xi_{\exp}(c)=e^c$, then $x_b(t)=e^{j2\pi b\,e^{t/t_r}}$ is a complex sinusoid with exponential IF law, $b\,\nu(t)=b\,e^{t/t_r}/t_r$, and the HOG-WD simplifies to:

$\text{HOG-WD}_{x_b}^{N,\xi_{\exp}}(t,\underline{\boldsymbol{f}}) = \lvert e^{t/t_r}\rvert^{N-1}\prod_{n=1}^{N-1}\delta(f_n+(-1)^n\frac{b}{t_r}e^{t/t_r})$.

The HOG-WD is also useful for analyzing impulse functions. Let the signal be an impulse function centered at time location t_0, $d(t)=\delta(t-t_0)$, then the HOG-WD is proportional to an impulse function localized at the same time t_0:

$$\text{HOG-WD}_d^{N,\xi}(t,\underline{\boldsymbol{f}})=\delta(t-t_0)\ \lvert t_r\nu(t_0)\rvert^{N-1}\ .$$

3.3. Some Special cases of the HOG-WD

Various interesting cases of the HOG-WD in (6), (7) and (9) can be obtained by appropriately selecting the function $\xi(c)$.

HO-WD: When $\xi(c)=\xi_{\text{linear}}(c)=c$ in (6), the HOG-WD simplifies to the HO-WD in (1), since $(\mathcal{W}_{\xi_{\text{linear}}}x)(t)=x(t)$.

HO-QD: When $\xi(c)=\xi_{\ln}(c)=\ln c$, the HOG-WD in (6) simplifies to a time/multi-frequency version[5] of the HO Q-distribution (HO-QD) in [9]. The HO-QD is given as

$$\text{HO-QD}_x^N(t,\underline{\boldsymbol{f}})=t_r^{N-1}\int_{\sigma_1}\cdots\int_{\sigma_{N-1}}M_x^{N,\xi_{\ln}}(t,\underline{\boldsymbol{\sigma}})\prod_{i=1}^{N-1}e^{-j2\pi t f_i\ln\sigma_i}\frac{d\sigma_i}{\sigma_i} \qquad (10)$$

the $(N-1)$th dimensional Mellin transform of a local Nth order scale moment function $M_x^{N,\xi_{\ln}}(t,\underline{\boldsymbol{\sigma}})$, that is defined in (8) with $\xi(c)=\xi_{\ln}(c)=\ln c$, and $\sigma_i\in\mathbb{R}^+$, $i=1,\cdots,N-1$ are scale lag parameters. The range of integration in (10) is from 0 to ∞.

HO-PWD: When $\xi(c)=\xi_\kappa(c)=\text{sgn}(c)\ \lvert c\rvert^\kappa$, $\kappa\neq 0$, the HOG-WD in (6) simplifies to the *new* HO power warped WD (HO-PWD):

$$\text{HO-PWD}_x^N(t,\underline{\boldsymbol{f}})=t_r^{N-1}\int_{\sigma_1}\cdots\int_{\sigma_{N-1}}M_x^{N,\xi_\kappa}(t,\underline{\boldsymbol{\sigma}})\prod_{i=1}^{N-1}e^{-j2\pi\frac{t_r f_i}{\xi'_\kappa(t)}\xi_\kappa(\sigma_i)}\lvert\xi'_\kappa(\sigma_i)\rvert d\sigma_i.$$

The HO-PWD is the $(N-1)$th dimensional κth power transform of the local Nth order κth power warped moment function in (8), $M_x^{N,\xi_\kappa}(t,\underline{\boldsymbol{\sigma}})$, with $\xi(c)=\xi_\kappa(c)$.

HO-EWD: When $\xi(c)=\xi_{\exp}(c)=e^c$, we obtain the *new* HO exponentially warped WD (HO-EWD). The HO-EWD is the $(N-1)$th dimensional exponential transform of the local Nth order exponential moment function, $M_x^{N,\xi_{\exp}}(t,\underline{\boldsymbol{\sigma}})$, which is defined in (8) with $\xi(c)=\xi_{\exp}(c)=e^c$,

$$\text{HO-EWD}_x^N(t,\underline{\boldsymbol{f}}) = t_r^{N-1}\int_{\sigma_1}\cdots\int_{\sigma_{N-1}}M_x^{N,\xi_{\exp}}(t,\underline{\boldsymbol{\sigma}})\prod_{i=1}^{N-1}e^{-j2\pi t_r f_i e^{\frac{-t}{t_r}}e^{\sigma_i}}e^{\sigma_i}d\sigma_i.$$

For example, the HO-EWD is useful in analyzing signals with exponential IF characteristics.

3.4. Desirable properties of the HOG-WD

The HOG-WD preserves many desirable signal properties important for time-frequency analysis, some of which are listed below.

[*P*-1] The HOG-WD satisfies the generalized warped time shift covariance property defined as:

$$y(t)=x(\eta(t,\tau))\Longrightarrow$$
$$\text{HOG-WD}_y^{N,\xi}(t,\underline{\boldsymbol{f}})=\text{HOG-WD}_x^{N,\xi}\Big(\eta(t,\tau),\frac{\nu(\eta(t,\tau))}{\nu(t)}\underline{\boldsymbol{f}}\Big), \qquad (11)$$

where $\eta(t,\tau)=t_r\xi^{-1}(\xi(\frac{t}{t_r})-\frac{\tau}{t_r})$. Note that when $\xi(c)=\xi_{\text{linear}}(c)=c$, the property in (11) simplifies to time-shift covariance : $y(t)=x(t-\tau)\Rightarrow\text{HOG-WD}_y^{N,\xi_{\text{linear}}}(t,\underline{\boldsymbol{f}})=\text{HO-WD}_x^N(t-\tau,\underline{\boldsymbol{f}})$. Furthermore, when $\xi(c)=\xi_{\ln}(c)=\ln(c)$, the property in (11) simplifies to scale covariance, i.e. $y(t)=x(te^{-\frac{\tau}{t_r}})\Rightarrow$

$$\text{HO-QD}_y^{N,\xi_{\ln}}(t,\underline{\boldsymbol{f}})=\text{HO-QD}_x^{N,\xi_{\ln}}(e^{-\frac{\tau}{t_r}}t,e^{\frac{\tau}{t_r}}\underline{\boldsymbol{f}}).$$

[*P*-2] For N even, and $\xi(c)$ given, the HOG-WD in (7) preserves alternating sign frequency shifts if changes in the signal's IF are proportional to $\nu(t)=\frac{d}{dt}\xi(\frac{t}{t_r})$:

$y(t)=x(t)e^{j2\pi b\xi(\frac{t}{t_r})}\Longrightarrow\text{HOG-WD}_y^{N,\xi}(t,f_1,f_2,\cdots,f_{N-1})=$
$\text{HOG-WD}_x^{N,\xi}(t,f_1-b\,\nu(t),f_2+b\,\nu(t),\cdots,f_{N-1}-b\,\nu(t))$.

The original HO-WD [4, 10] did not satisfy any frequency shift covariance property for $N>2$, since the local HO moment function in [4] was defined with only one conjugated signal term. By

[5] The HO-QD in [9] is a time/multi-scale representation, $\mathrm{T}_x^N(t,\underline{\boldsymbol{c}})$. They are related as $\text{HO-QD}_x^N(t,\underline{\boldsymbol{f}})=t_r^{N-1}\mathrm{T}_x^N(t,\underline{\boldsymbol{c}})\mid_{\underline{\boldsymbol{c}}=\mathrm{t}\,\underline{\boldsymbol{f}}}$.

alternately conjugating the signal terms using $\mathcal{L}_n$ in (8) in a manner similar to [9], we obtain a HO-WD with $\xi(c)$=c in (7) which satisfies this property.

[$\boldsymbol{P}$-3] The HOG-WD preserves generalized warped scalings on a signal: $y(t) = x(\mu(t,a)) \Rightarrow$

HOG-WD$_y^{N,\xi}(t, \underline{\boldsymbol{f}}) = |a|^{1-N}$HOG-WD$_x^{N,\xi}\left(\mu(t,a), \frac{\nu(\mu(t,a))}{a\nu(t)}\underline{\boldsymbol{f}}\right)$, where $\mu(t,a) = t_r\xi^{-1}(a\xi(\frac{t}{t_r}))$. This property simplifies to scale covariance when $\xi(c) = c$, or $\xi(c) = \xi_\kappa(c)$, and to constant time-shift covariance when $\xi(c) = e^c$.

[$\boldsymbol{P}$-4] The HO-WD in [4] was designed to preserve the IF property, i.e. the mean frequency of the HO-WD at a given time yields the derivative of the phase of the signal. By choosing the centering parameter $\alpha = \frac{1}{N}\sum_{l=1}^{N-1}\tau_l$, in the HO moment function in (1), the HO-WD satisfies the centering constraint [5, 4], necessary for preserving the IF property. Since the HOG-WD is a warped version of the HO-WD, it preserves a generalized IF property. Specifically, if $x(t) = r(t)\, e^{j\varphi(t)}, r(t) \geq 0$, then the generalized IF (mean conditional frequency) property is given as:

$$\frac{\int_{f_1}\cdots\int_{f_{N-1}} f_i \text{ HOG-WD}_x^{N,\xi}(t, \underline{\boldsymbol{f}})\, df_1\cdots df_{N-1}}{\int_{f_1}\cdots\int_{f_{N-1}}\text{HOG-WD}_x^{N,\xi}(t, \underline{\boldsymbol{f}})\, df_1\cdots df_{N-1}} = \frac{\lambda_i}{2\pi}\frac{d}{dt}\varphi(t)$$

where $\lambda_i = (-1)^{i+1}$ for N even and $\lambda_i = (-1)^{i+1} + \frac{1}{N}$ for N odd.

[$\boldsymbol{P}$-5] Multiple integration over all frequency parameters of the Nth HOG-WD yields the instantaneous Nth order temporal moment,

$$\begin{aligned} &|\xi'(\frac{t}{t_r})|^{1-N} \int_{f_1}\cdots\int_{f_{N-1}}\text{HOG-WD}_x^{N,\xi}(t, \underline{\boldsymbol{f}})\; df_1\cdots df_{N-1} \\ &= x^*(t)\prod_{n=1}^{N-1}(\mathcal{L}_n x)(t) \;=\; M_x^{N,\xi}\left(t, \xi^{-1}(0), \cdots, \xi^{-1}(0)\right). \end{aligned}$$

[$\boldsymbol{P}$-6] Integrating the HOG-WD along curves that are proportional to the IF, $c_i\,\nu(t), i = 1, \cdots, N-1$, yields the HOG moment spectrum, $P_x^{N,\xi}(\underline{\boldsymbol{\gamma}})$:

$$\begin{aligned} P_x^{N,\xi}(\underline{\boldsymbol{\gamma}}) &= \int_t \text{HOG-WD}_x^{N,\xi}(t, c_1\,\nu(t), \cdots, c_{N-1}\,\nu(t))\,|t_r\nu(t)|\,dt \\ &= \rho_x^*\Big(\sum_{i=1}^{N-1}\gamma_i\Big)\prod_{n=1}^{N-1}(\mathcal{L}_n\rho_x)((-1)^{n+1}\gamma_n) \end{aligned}$$

where $\rho_x(c)$ is the 1-D generalized transform in (5). When $N = 3$, the HOG spectrum simplifies to the bispectrum in [7, 6] for $\xi(c) = c$, the scale bispectrum in [9] for $\xi(c) = \ln c$, the *new* power warped bispectrum for $\xi(c) = \xi_\kappa(c)$, and the *new* exponential bispectrum for $\xi(c) = e^c$. The HOG moment function for a deterministic signal, $p_x^{N,\xi}(\underline{\boldsymbol{\sigma}}) = \int t_r^N M_x^{N,\xi}(t, \underline{\boldsymbol{\sigma}})\,|\nu(t)|\;dt$, and the HOG spectrum, $P_x^{N,\xi}(\underline{\boldsymbol{\gamma}})$, share the same generalized transform relation as for random signals in (4).

4. GENERALIZED HIGHER ORDER CLASS

In [4], Cohen's kernel function approach [3] was applied to the HO-WD to obtain an entire class of higher order TFRs:

$$C_x^N(t, \underline{\boldsymbol{f}}) = \iint_{\hat{f}_1}\cdots\int_{\hat{f}_{N-1}}\psi_C^N(t - \hat{t}, \underline{\boldsymbol{f}} - \underline{\hat{\boldsymbol{f}}})\text{ HO-WD}_x^N(\hat{t}, \underline{\hat{\boldsymbol{f}}})\; d\hat{t}\prod_{i=1}^{N-1} df_i.$$

By choosing the kernel, ψ_C^N, to be a low-pass function, cross terms in the non-linear HO-WD can be attenuated. In the same spirit, we derived a class of HO generalized TFRs by applying the warping procedure in (6) to the HO Cohen's class above:

$$\begin{aligned} G_x^N(t, \underline{\boldsymbol{f}}) = \int\int_{\hat{f}_1}\cdots\int_{\hat{f}_{N-1}} \psi_G^N\left(\xi(\frac{t}{t_r}) - \xi(\frac{\hat{t}}{t_r}), \frac{\underline{\boldsymbol{f}}}{\nu(t)} - \frac{\underline{\hat{\boldsymbol{f}}}}{\nu(\hat{t})}\right) \\ \cdot\text{HOG-WD}_x^{N,\xi}(\hat{t}, \underline{\hat{\boldsymbol{f}}})\;\frac{d\hat{t}}{|\nu(\hat{t})|^{N-2}}\prod_{n=1}^{N-1} df_n. \end{aligned}$$

This *new* HO generalized warped class consists of TFRs that always satisfy the generalized frequency-shift covariance property for N even. Moreover, the kernel ψ_G^N of the HO-TFR can be chosen to reduce the troublesome cross-terms of the HOG-WD in various applications.

5. CONCLUSION

We have provided a unifying framework for the concepts of higher order (HO) moment functions, their associated transform pairs, the HO spectra, and HO TFRs. They can be important for the analysis of random and deterministic signals with arbitrary, dispersive IF characteristics. We have shown how our work generalizes previous TFRs, as for special cases it simplifies to the HO-WD in [4], the HO-QD in [9], and to the GWWD in [8]. It also generalizes previous spectra: the bispectrum in [7], the scale bispectrum in [9], and the generalized power spectrum in [8], while introducing the *new* power warped bispectrum and exponential bispectrum. We demonstrated the importance of our new results by showing, for example, that the new power warped bispectrum is well-suited for the detection of phase coupled power sinusoids.

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