# WAVELET SYSTEMS WITH ZERO MOMENTS\*

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## ABSTRACT

The Coifman wavelets created by Daubechies have more zero moments than imposed by specifications. This results in systems with approximately equal numbers of zero scaling function and wavelet moments and gives a partitioning of the systems into three well defined classes. The nonunique solutions are more complex than for Daubechies wavelets.

## Introduction

Wavelet systems have proven to be a popular and effective basis system for the expansion and representation of signals and images [1, 2]. In addition to the requirements of multiresolution analysis, orthogonality, and finite support, requiring that wavelet moments vanish has also proven valuable. Coifman has suggested that also requiring scaling function moments to be zero has some advantages. Daubechies created the "coiflet" system by setting an equal number of scaling function and wavelet moments to zero [3, 2, 4]. This number of required zero moments plus one is called the degree of the coiflet system.

It has been observed that the even degree Coifman systems always achieve one more zero scaling function moment than is specified and the odd degree systems also had one more zero scaling function moment, but it was not contiguous with the specified ones. We also discovered that certain lengths of scaling function coefficient vectors were never generated by the usual approach of looking at minimum length filters at each degree.

In this paper we examine the characteristics of coiflet systems that do not have exactly the same number of scaling function and wavelet zero moments. This results in three classes of coiflet systems, each with well defined characteristics. Two of these had been previously reported and one is new. We define these classes and describe their characteristics. We also show why the total number of zero moments is larger than that predicted by the available degrees of freedom for coiflets while the Daubechies wavelet systems achieve exactly the number predicted by the degrees of freedom. Finally, we note that the solutions that satisfy the requirements are not unique and some, despite zero wavelet moments, are very rough. Likewise, despite zero scaling function moments, some are not at all symmetric.

# Zero Moment Wavelet Systems

The scaling function  $\varphi(t)$  which generates a traditional wavelet system [1, 2] is defined as the solution to the mul-

tiresolution analysis (MRA) equation

$$\varphi(t) = \sum_{n} \sqrt{2} h(n) \varphi(2t - n) \tag{1}$$

and these functions are usually required to be orthogonal over integer shifts by  $\int \varphi(t) \varphi(t-k) dt = \delta(k)$  where *n* and *k* are integers. It has been shown [2, 1] that for (1) to have a solution and be orthogonal,

$$\sum_{n} h(n) = \sqrt{2} \text{ and } \sum_{n} h(n) h(n-2k) = \delta(k).$$
 (2)

We call the support N, the length of the scaling vector. That is,  $h = \{h(0), h(1), \dots, h(N-1)\}.$ 

The problem of wavelet system design is to find N values for h(n) that satisfy the one linear equation in (2) and the N/2quadratic equations in (2). That leaves N/2 - 1 degrees of freedom in designing a wavelet system for some particular application.

#### The Daubechies Systems

Daubechies [2] uses these degrees of freedom to require N/2 - 1 wavelet moments to be zero. This means

$$m_1(k) = \int t^k \psi(t) dt = 0 \text{ for } k = 1, 2, 3, \cdots, K$$
 (3)

where K = N/2 - 1. One can show [5] that this is true if and only if

$$\mu_1(k) = \sum_n n^k h_1(n) = 0 \text{ for } k = 1, 2, 3, \cdots, K$$
 (4)

where  $h_1(n)$  are the wavelet coefficients which define the wavelet  $\psi(t)$  by

$$\psi(t) = \sum_{n} \sqrt{2} h_1(n) \varphi(2t - n).$$
(5)

For the wavelet to be orthogonal to the scaling function requires

$$h_1(n) = (-1)^n h(1-n)$$
 and  $\sum_n h_1(n) = \mu_1(0) = 0$ 
  
(6)

Similar definitions of the moments for the scaling function and scaling coefficients are given by

$$m(k) = \int t^k \varphi(t) dt$$
 and  $\mu(k) = \sum_n n^k h(n)$ . (7)

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Daubechies developed a very nice analytical method for designing the h(n) that satisfies (2) and (4). Three reasons for requiring the maximum number of zero wavelet moments are to obtain smooth scaling functions, to represent certain order polynomials exactly by shifted scaling functions, and to have a tractable design method. The characteristic of the Daubechies wavelet system that is important to this paper is that the Daubechies wavelets have exactly the same number of zero wavelet moments as degrees of freedom, no more and no less.

## The Coifman Systems

Coifman suggested setting both scaling function and wavelet moments to zero to obtain more symmetry and compactness for numerical analysis applications. In addition to (3), Daubechies [4, 2] posed the requirements

$$m(k) = \int t^k \varphi(t) dt = 0 \text{ for } k = 1, 2, \cdots, L - 1$$
 (8)

and one can show [5] that this is true if and only if

$$\mu(k) = \sum_{n} n^{k} h(n) = 0. \text{ for } k = 1, 2, \cdots, L - 1$$
 (9)

From (2) we see that  $\mu(0) = \sqrt{2}$  and cannot be made zero and (6) requires  $\mu_1(0) = 0$ . Solutions to (1) that also satisfy both (3) and (8) require solutions to (2) and (2) that also satisfy (4) and (9). The wavelet system generated from these solutions with K = L - 1 are called coiflets of degree L. The basis functions are often more symmetric than the Daubechies ones, but less smooth. In the design of these coiflets, one obtains more total zero moments than N/2 - 1. This was first noted by Beylkin, et al [3].

This definition imposes the requirement that there be at least L-1 zero scaling function moments and at least L-1 wavelet moments in addition to the one zero moment of  $m_1(0)$  required by orthogonality. This system is said to be of order or degree L and sometime has the additional requirement that the length of the scaling function filter h(n), which is denoted N, is minimum [2, 4].

The length-4 wavelet system has only one degree of freedom, so cannot have both a scaling function moment and wavelet moment be zero (see Table 1). Tian [6] has derived formulas for four length-6 coiflets. These are:

$$h = \left[\frac{-3 \pm A}{16\sqrt{2}}, \frac{1 \mp A}{16\sqrt{2}}, \frac{7 \mp A}{8\sqrt{2}}, \frac{7 \pm A}{8\sqrt{2}}, \frac{5 \pm A}{16\sqrt{2}}, \frac{1 \mp A}{16\sqrt{2}}\right],\tag{10}$$

or

$$h = \left[\frac{-3 \pm B}{16\sqrt{2}}, \frac{1 \mp B}{16\sqrt{2}}, \frac{3 \mp B}{8\sqrt{2}}, \frac{3 \pm B}{8\sqrt{2}}, \frac{13 \pm B}{16\sqrt{2}}, \frac{9 \mp B}{16\sqrt{2}}\right],\tag{11}$$

where  $A = \sqrt{7}$ ,  $B = \sqrt{15}$  with the first formula (10) giving the same result as Daubechies [2, 4] (corrected) and that of Odegard and the third giving the same result as Wickerhauser [7]. The results from (10) are included in Table 1 along with the discrete moments of the scaling function and wavelet,  $\mu(k)$ and  $\mu_1(k)$  for k = 0, 1, 2, 3. The design of a length-6 Coifman system specifies one zero scaling function moment and one zero wavelet moment (in addition to  $\mu_1(0) = 0$ ) but we, in fact, obtain one extra zero scaling function moment. That is the result of  $m(2) = m(1)^2$  from [5]. In other words, we get one more zero scaling function moment than the two degrees of freedom would seem to indicate. This is true for all lengths  $N = 6\ell$  for  $\ell = 1, 2, 3, \cdots$  and is a result of an interaction between the scaling function moments and the wavelet moments described later.

The scaling function from (10) is fairly symmetric, but not around its center and the other three designs in (10), (11), and (11) are not symmetric at all. The scaling functions from (10) and (11) are fairly smooth but the one from (10) is very rough and from (11) seems to be fractal. Examination of the frequency response  $H(\omega)$  shown in Figure 2 for the FIR filters h(n) shows very similar frequency responses for (10) and (11) with (10) having a somewhat irregular but monotonic frequency response and (11) having a zero on the unit circle at  $\omega = \pi/3$ , i.e., not satisfying Cohen's condition [2] for an orthogonal basis. These four designs, all satisfying the same necessary conditions, have very different characteristics. This tells us to be careful in using zero moment methods to design wavelet systems. The designs are not unique and some are better than others.

Table 1: Coiflet Scaling Function and Wavelet Coefficients plus their Discrete Moments

	N = 6,	Deg. $L = 2$		
n	h(n)	$h_1(n)$	$\mu(k)  \mu_1(k)$	k
-2	-0.0727326	0.0156557	1.4142 0	0
-1	0.3378976	-0.0727326	0 0	1
0	0.8525720	-0.3848648	0 -1.163	2
1	0.3848648	0.8525720	-0.3757 -3.866	3
2	-0.0727326	-0.3378976	-2.8727-10.267	4
3	-0.0156557	-0.0727326		-
-	N-8	Deg $L = 3$		
-4	10 - 0,	0.0156557	1.414 0	0
-4	0.0408750	0.0130337	1.414 0	1
-5	0.1406250	0.3848648		2
-2	-0.1400230	1 3848648	2004 018	2
-1	1 38/86/8	0.4384804	-2.334 0.18	1
1	0.3848648	-0.4384804	-45 851 -43 97	5
2	0.0727326	-0.1400230	63 630 271 34	5
2	-0.0727320	0.0211001	05.059 271.54	0
3	-0.0130337	0.0408730		
	N = 12,	Deg. $L = 4$	1.41 0	
-4	0.01638/3	0.0007205	1.41 0	0
-3	-0.0414649	0.0018232	0 0	1
-2	-0.06/3/25	-0.0056114	0 0	2
-1	0.3861100	-0.0236801	0  0	3
0	0.8127236	0.0594344	0 11	4
1	0.4170051	0.0764885	5 01 175	5
-	0.4170051	0.0704005	-3.91 173	5
2	-0.0764885	-0.4170051	0 1795	6
2 3	-0.0764885 -0.0594344	-0.4170051 -0.8127236	-5.91 175 0 1795 -586.34 15230	6 7
2 3 4	-0.0764885 -0.0594344 0.0236801	-0.4170051 -0.8127236 -0.3861100	-5.91 175 0 1795 -586.34 15230 3096.31117752	6 7 8
2 3 4 5	-0.0764885 -0.0594344 0.0236801 0.0056114	-0.4170051 -0.8127236 -0.3861100 0.0673725	-5.91 173 0 1795 -586.34 15230 3096.31117752	6 7 8
2 3 4 5 6	-0.0764885 -0.0594344 0.0236801 0.0056114 -0.0018232	$\begin{array}{c} -0.4170051 \\ -0.8127236 \\ -0.3861100 \\ 0.0673725 \\ 0.0414649 \end{array}$	-5:91 175 0 1795 -586.34 15230 3096.31117752	6 7 8
2 3 4 5 6 7	-0.0764885 -0.0594344 0.0236801 0.0056114 -0.0018232 -0.0007205	$\begin{array}{c} -0.4170051\\ -0.8127236\\ -0.3861100\\ 0.0673725\\ 0.0414649\\ -0.0163873 \end{array}$	-586.34 15230 3096.31117752	6 7 8
2 3 4 5 6 7	$\begin{array}{c} -0.0764885\\ -0.0594344\\ 0.0236801\\ 0.0056114\\ -0.0018232\\ -0.0007205\\ \hline N=4 \end{array}$	-0.4170051 -0.8127236 -0.3861100 0.0673725 0.0414649 -0.0163873 Deg $L = 1$	-53.51 175 0 1795 -586.34 15230 3096.31117752	6 7 8
$ \begin{array}{c} 2\\ 3\\ 4\\ 5\\ 6\\ 7\\ \hline n \end{array} $	$\begin{array}{c} -0.0764885\\ -0.0594344\\ 0.0236801\\ 0.0056114\\ -0.0018232\\ -0.0007205\\ \hline N=4,\\ h(n) \end{array}$	-0.4170051 -0.4170051 -0.8127236 -0.3861100 0.0673725 0.0414649 -0.0163873 Deg. $L = 1$ $h_1(n)$	-5.51 - 175 0 - 1795 -586.34 - 15230 3096.31 - 117752 $\mu(k) - \mu_1(k)$	6 7 8
$ \begin{array}{c} 2\\ 3\\ 4\\ 5\\ 6\\ 7\\ \hline n\\ -1\\ \end{array} $	$\begin{array}{c} 0.0764885\\ -0.0594344\\ 0.0236801\\ 0.0056114\\ -0.0018232\\ -0.0007205\\ \hline N=4,\\ h(n)\\ 0.224138\\ \end{array}$	$\begin{array}{c} -0.4170051\\ -0.4170051\\ -0.8127236\\ -0.3861100\\ 0.0673725\\ 0.0414649\\ -0.0163873\\ \hline \text{Deg. } L=1\\ \hline h_1(n)\\ 0.1294095\\ \end{array}$	$\begin{array}{c} -3.31 & 173 \\ 0 & 1795 \\ -586.34 & 15230 \\ 3096.31 & 117752 \\ \hline \\ \mu(k) & \mu_1(k) \\ 1 & 414 & 0 \end{array}$	6 7 8 <i>k</i>
$ \begin{array}{c} 2\\ 3\\ 4\\ 5\\ 6\\ 7\\ \hline \\ n\\ -1\\ 0\\ \end{array} $	$\begin{array}{c} 0.0764885\\ -0.0594344\\ 0.0236801\\ 0.0056114\\ -0.0018232\\ -0.0007205\\ \hline N=4,\\ h(n)\\ 0.2241438\\ 0.8365163\\ \end{array}$	$\begin{array}{c} \text{-0.4170051} \\ \text{-0.4170051} \\ \text{-0.8127236} \\ \text{-0.3861100} \\ \text{0.0673725} \\ \text{0.0414649} \\ \text{-0.0163873} \\ \hline \\ $	$\begin{array}{c} -3.31 & 173 \\ 0 & 1795 \\ -586.34 & 15230 \\ 3096.31 & 117752 \\ \hline \\ \mu(k) & \mu_1(k) \\ 1.414 & 0 \\ 0 & -0 & 517 \\ \end{array}$	5 6 7 8 <i>k</i> 0 1
	$\begin{array}{c} 0.0764885\\ -0.0594344\\ 0.0236801\\ 0.0056114\\ -0.0018232\\ -0.0007205\\ \hline N=4,\\ h(n)\\ 0.2241438\\ 0.8365163\\ 0.4829629\\ \end{array}$	$\begin{array}{c} 0.3164033\\ -0.4170051\\ -0.8127236\\ -0.3861100\\ 0.0673725\\ 0.0414649\\ -0.0163873\\ \hline \\ \hline$	$\begin{array}{c} -5.31 & 173 \\ 0 & 1795 \\ -586.34 & 15230 \\ 3096.31 & 117752 \\ \hline \mu(k) & \mu_1(k) \\ \hline 1.414 & 0 \\ 0 & 0.517 \\ 0 & 189 & 0 & 189 \\ \end{array}$	5 6 7 8 8 <i>k</i> 0 1 2
$ \begin{array}{c} 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ \hline n \\ -1 \\ 0 \\ 1 \\ 2 \\ \end{array} $	$\begin{array}{c} 0.0764885\\ -0.0594344\\ 0.0236801\\ 0.0056114\\ -0.0018232\\ -0.0007205\\ \hline N=4,\\ h(n)\\ 0.2241438\\ 0.8365163\\ 0.4829629\\ -0.1294095\\ \end{array}$	$\begin{array}{c} \text{-0.4170051} \\ \text{-0.4170051} \\ \text{-0.8127236} \\ \text{-0.3861100} \\ \text{0.0673725} \\ \text{0.0414649} \\ \text{-0.0163873} \\ \hline \\ $	$\begin{array}{c} -3.31 & 173 \\ 0 & 1795 \\ -586.34 & 15230 \\ 3096.31 & 117752 \\ \hline \mu(k) & \mu_1(k) \\ \hline 1.414 & 0 \\ 0 & -0.517 \\ 0.189 & 0.189 \\ -0.776 & 0.827 \\ \end{array}$	5 6 7 8 <i>k</i> 0 1 2 3
$     \begin{array}{c}       2 \\       3 \\       4 \\       5 \\       6 \\       7 \\       \hline       n \\       -1 \\       0 \\       1 \\       2 \\       \end{array}   $	$\begin{array}{c} 0.0764885\\ -0.0594344\\ 0.0236801\\ 0.0056114\\ -0.0018232\\ -0.0007205\\ \hline N=4,\\ \hline h(n)\\ 0.2241438\\ 0.8365163\\ 0.4829629\\ -0.1294095\\ \hline N=10 \end{array}$	$\begin{array}{c} 0.316030\\ -0.4170051\\ -0.8127236\\ -0.3861100\\ 0.0673725\\ 0.0414649\\ -0.0163873\\ \hline \text{Deg. } L=1\\ \hline h_1(n)\\ \hline 0.1294095\\ 0.4829629\\ -0.8365163\\ 0.2241438\\ \hline \text{Deg. } L=2\\ \end{array}$	$\begin{array}{c} -5.31 & 173 \\ 0 & 1795 \\ -586.34 & 15230 \\ 3096.31 & 117752 \\ \hline \\ \mu(k) & \mu_1(k) \\ \hline 1.414 & 0 \\ 0 & -0.517 \\ 0.189 & 0.189 \\ -0.776 & 0.827 \\ \end{array}$	6 7 8 <i>k</i> 0 1 2 3
$     \begin{array}{c}       2 \\       3 \\       4 \\       5 \\       6 \\       7 \\       \hline       n \\       -1 \\       0 \\       1 \\       2 \\       \hline       2       \end{array} $	$\begin{array}{c} 0.0764885\\ -0.0594344\\ 0.0236801\\ 0.0056114\\ -0.0018232\\ -0.0007205\\ \hline N=4,\\ h(n)\\ \hline 0.2241438\\ 0.8365163\\ 0.4829629\\ -0.1294095\\ \hline N=10,\\ 0.021294\end{array}$	$\begin{array}{c} \text{-0.4170051} \\ \text{-0.4170051} \\ \text{-0.8127236} \\ \text{-0.3861100} \\ \text{0.0673725} \\ \text{0.0414649} \\ \text{-0.0163873} \\ \hline \\ \hline \\ \text{Deg. } L = 1 \\ \hline \\ \hline \\ h_1(n) \\ \text{0.1294095} \\ \text{0.4829629} \\ \text{-0.8365163} \\ \text{0.2241438} \\ \hline \\ \hline \\ \\ \text{Deg. } L = 3 \\ \hline \\$	$\begin{array}{c} -5.31 & 173 \\ 0 & 1795 \\ -586.34 & 15230 \\ 3096.31 & 117752 \\ \hline \\ \mu(k) & \mu_1(k) \\ 1.414 & 0 \\ 0 & -0.517 \\ 0.189 & 0.189 \\ -0.776 & 0.827 \\ \hline \\ 1.41 & 0 \\ \hline \end{array}$	6 7 8 <i>k</i> 0 1 2 3
	$\begin{array}{c} 0.0764885\\ -0.0594344\\ 0.0236801\\ 0.0056114\\ -0.0018232\\ -0.0007205\\ \hline N=4,\\ h(n)\\ 0.2241438\\ 0.8365163\\ 0.4829629\\ -0.1294095\\ \hline N=10,\\ 0.0321284\\ 0.075302\\ \hline \end{array}$	$\begin{array}{c} \text{-0.4170051} \\ \text{-0.4170051} \\ \text{-0.8127236} \\ \text{-0.3861100} \\ \text{0.0673725} \\ \text{0.0414649} \\ \text{-0.0163873} \\ \hline \text{Deg. } L = 1 \\ \hline h_1(n) \\ \text{0.1294095} \\ \text{0.4829629} \\ \text{-0.8365163} \\ \text{0.2241438} \\ \hline \text{Deg. } L = 3 \\ \hline 0.0002337 \\ \text{0.0002337} \\ \hline 0.00025496 \end{array}$	$\begin{array}{c} -3.31 & 173 \\ 0 & 1795 \\ -586.34 & 15230 \\ 3096.31 & 117752 \\ \hline \\ \mu(k) & \mu_1(k) \\ 1.414 & 0 \\ 0 & -0.517 \\ 0.189 & 0.189 \\ -0.776 & 0.827 \\ \hline \\ 1.41 & 0 \\ 0 & 0 \\ \end{array}$	6 7 8 <i>k</i> 0 1 2 3
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$ \begin{array}{c} 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ \hline n \\ -1 \\ 0 \\ 1 \\ 2 \\ \hline -2 \\ -1 \\ 0 \\ 1 \\ 0 \\ 1 \\ 2 \\ \hline \end{array} $	$\begin{array}{c} 0.0764885\\ -0.0594344\\ 0.0236801\\ 0.0056114\\ -0.0018232\\ -0.0007205\\ \hline N=4,\\ h(n)\\ 0.2241438\\ 0.8365163\\ 0.4829629\\ -0.1294095\\ \hline N=10,\\ 0.0321284\\ -0.0755392\\ -0.0969350\\ 0.4915400\\ \end{array}$	$\begin{array}{c} \text{-0.4170051} \\ \text{-0.4170051} \\ \text{-0.8127236} \\ \text{-0.3861100} \\ \text{0.0673725} \\ \text{0.0414649} \\ \text{-0.0163873} \\ \hline \\ $	$\begin{array}{c} -3.31 & 17.3 \\ 0 & 1795 \\ -586.34 & 15230 \\ 3096.31 & 117752 \\ \hline \\ \mu(k) & \mu_1(k) \\ \hline \\ 1.414 & 0 \\ 0 & -0.517 \\ 0.189 & 0.189 \\ -0.776 & 0.827 \\ \hline \\ \hline \\ 1.41 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ \end{array}$	
$ \begin{array}{c} 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ \hline n \\ -1 \\ 0 \\ 1 \\ 2 \\ \hline -2 \\ -1 \\ 0 \\ 1 \\ 2 \\ \hline \end{array} $	$\begin{array}{c} 0.0764885\\ -0.0594344\\ 0.0236801\\ 0.0056114\\ -0.0018232\\ -0.0007205\\ \hline N=4,\\ h(n)\\ 0.2241438\\ 0.8365163\\ 0.4829629\\ -0.1294095\\ \hline N=10,\\ 0.0321284\\ -0.0755392\\ -0.0969350\\ 0.4915490\\ 0.8051410\\ \end{array}$	$\begin{array}{c} \text{-0.4170051} \\ \text{-0.4170051} \\ \text{-0.8127236} \\ \text{-0.3861100} \\ \text{0.0673725} \\ \text{0.0414649} \\ \text{-0.0163873} \\ \hline \\ $	$\begin{array}{c} -5.31 & 173 \\ 0 & 1795 \\ -586.34 & 15230 \\ 3096.31 & 117752 \\ \hline \\ \mu(k) & \mu_1(k) \\ 1.414 & 0 \\ 0 & 0 & 0.189 \\ -0.776 & 0.827 \\ \hline \\ 1.41 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 3 \\ 0 & 24 \\ \end{array}$	6 7 8 <i>k</i> 0 1 2 3 0 1 2 3 4
$ \begin{array}{c} 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ \hline \\ -1 \\ 0 \\ 1 \\ 2 \\ \hline \\ -2 \\ -1 \\ 0 \\ 1 \\ 2 \\ 3 \\ \end{array} $	$\begin{array}{c} 0.0764885\\ -0.0594344\\ 0.0236801\\ 0.0056114\\ -0.0018232\\ -0.0007205\\ \hline N=4,\\ h(n)\\ 0.2241438\\ 0.8365163\\ 0.4829629\\ -0.1294095\\ \hline N=10,\\ 0.0321284\\ -0.0755392\\ -0.0969350\\ 0.4915490\\ 0.8051410\\ 0.3044125\\ \hline \end{array}$	$\begin{array}{c} 0.3170051\\ -0.4170051\\ -0.8127236\\ -0.3861100\\ 0.0673725\\ 0.0414649\\ -0.0163873\\ \hline \\ \hline$	$\begin{array}{c} -5.31 & 173 \\ 0 & 1795 \\ -586.34 & 15230 \\ 3096.31 & 117752 \\ \hline \\ \mu(k) & \mu_1(k) \\ \hline 1.414 & 0 \\ 0 & -0.517 \\ 0.189 & 0.189 \\ -0.776 & 0.827 \\ \hline \\ \hline \\ 1.41 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 3 \\ 0 & 24 \\ 14.70 & 128 \\ \end{array}$	6 7 8 <i>k</i> 0 1 2 3 4 5
$ \begin{array}{c} 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ \hline n \\ -1 \\ 0 \\ 1 \\ 2 \\ \hline -2 \\ -1 \\ 0 \\ 1 \\ 2 \\ 3 \\ 4 \\ \hline \end{array} $	$\begin{array}{c} 0.0764885\\ -0.0594344\\ 0.0236801\\ 0.0056114\\ -0.0018232\\ -0.0007205\\ \hline N=4,\\ h(n)\\ 0.2241438\\ 0.8365163\\ 0.4829629\\ -0.1294095\\ \hline N=10,\\ 0.0321284\\ -0.0755392\\ -0.0969350\\ 0.4915490\\ 0.8051410\\ 0.3044135\\ 0.0324772\\ \end{array}$	$\begin{array}{c} \text{-0.4170051} \\ \text{-0.4170051} \\ \text{-0.8127236} \\ \text{-0.3861100} \\ \text{0.0673725} \\ \text{0.0414649} \\ \text{-0.0163873} \\ \hline \\ $	$\begin{array}{c} -3.31 & 173 \\ 0 & 1795 \\ -586.34 & 15230 \\ 3096.31 & 117752 \\ \hline \\ \mu(k) & \mu_1(k) \\ 1.414 & 0 \\ 0 & -0.517 \\ 0.189 & 0.189 \\ -0.776 & 0.827 \\ \hline \\ 1.41 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 3 \\ 0 & 24 \\ -14.70 & 138 \\ 64.98 & 710 \\ \hline \end{array}$	k         0         1         2         3         0         1         2         3         4         5         6         6         7         8         10         11         12         3         4         5         6         6         6         6         10         10         10         10         10         10         10         10         10 <th10< th=""> <th10< th=""> <th10< th="">         &lt;</th10<></th10<></th10<>
$ \begin{array}{c} 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ \hline n \\ -1 \\ 0 \\ 1 \\ 2 \\ \hline -2 \\ -1 \\ 0 \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ \hline \end{array} $	$\begin{array}{c} 0.0764885\\ -0.0594344\\ 0.0236801\\ 0.0056114\\ -0.0018232\\ -0.0007205\\ \hline N=4,\\ h(n)\\ 0.2241438\\ 0.8365163\\ 0.4829629\\ -0.1294095\\ \hline N=10,\\ 0.0321284\\ -0.0755392\\ -0.0969350\\ 0.4915490\\ 0.8051410\\ 0.3044135\\ -0.0337773\\ -0.0337773\\ 0.0135532\\ \end{array}$	$\begin{array}{c} \text{-0.4170051} \\ \text{-0.4170051} \\ \text{-0.8127236} \\ \text{-0.3861100} \\ \text{0.0673725} \\ \text{0.0414649} \\ \text{-0.0163873} \\ \hline \text{Deg. } L = 1 \\ \hline h_1(n) \\ \text{0.1294095} \\ \text{0.4829629} \\ \text{-0.8365163} \\ \text{0.2241438} \\ \hline \text{Deg. } L = 3 \\ \hline \text{0.0002337} \\ \text{-0.0005496} \\ \text{-0.0135503} \\ \text{0.0337773} \\ \text{0.3044135} \\ \text{-0.8051410} \\ \text{0.4915490} \\ \text{0.969350} \\ \hline \end{array}$	$\begin{array}{c} -3.31 & 173 \\ 0 & 1795 \\ -586.34 & 15230 \\ 3096.31 & 117752 \\ \hline \\ \mu(k) & \mu_1(k) \\ 1.414 & 0 \\ 0 & -0.517 \\ 0.189 & 0.189 \\ -0.776 & 0.827 \\ \hline \\ 1.41 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 3 \\ 0 & 24 \\ -14.70 & 138 \\ 64.98 & 710 \\ \hline \end{array}$	
$ \begin{array}{c} 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ \hline n \\ -1 \\ 0 \\ 1 \\ 2 \\ -2 \\ -1 \\ 0 \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ \end{array} $	$\begin{array}{c} 0.0764885\\ -0.0594344\\ 0.0236801\\ 0.0056114\\ -0.0018232\\ -0.0007205\\ \hline N=4,\\ h(n)\\ 0.2241438\\ 0.8365163\\ 0.4829629\\ -0.1294095\\ \hline N=10,\\ 0.0321284\\ -0.0755392\\ -0.0969350\\ 0.4915490\\ 0.8051410\\ 0.3044135\\ -0.0337773\\ -0.0135503\\ -0.0135503\\ -0.035036\\ -0.0135503\\ -0.015503\\ -0.015503\\ -0.015503\\ -0.015503\\ -0.015503\\ -0.015503\\ -0.015503\\ -0.015503\\ -0.015503\\ -0.015503\\ -0.01550\\ -0.0150\\ -0.01550\\ -0.0$	$\begin{array}{c} \text{-0.4170051} \\ \text{-0.4170051} \\ \text{-0.8127236} \\ \text{-0.3861100} \\ \text{0.0673725} \\ \text{0.0414649} \\ \text{-0.0163873} \\ \hline \\ $	$\begin{array}{c} -3.31 & 173 \\ 0 & 1795 \\ -586.34 & 15230 \\ 3096.31 & 117752 \\ \hline \\ \mu(k) & \mu_1(k) \\ 0 & 0 & -0.517 \\ 0.189 & 0.189 \\ -0.776 & 0.827 \\ \hline \\ 1.41 & 0 \\ 0 & $	6 7 8 <i>k</i> 0 1 2 3 4 5 6
$ \begin{array}{c} 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ \hline n \\ -1 \\ 0 \\ 1 \\ 2 \\ -2 \\ -1 \\ 0 \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ \hline \end{array} $	$\begin{array}{c} 0.0764885\\ -0.0594344\\ 0.0236801\\ 0.0056114\\ -0.0018232\\ -0.0007205\\ \hline $N=4$,\\ $h(n)$\\ 0.2241438\\ 0.8365163\\ 0.4829629\\ -0.1294095\\ \hline $N=10$,\\ 0.0321284\\ -0.0755392\\ -0.0969350\\ 0.4915490\\ 0.8051410\\ 0.3044135\\ -0.0337773\\ -0.0135503\\ 0.0005496\\ 0.0005496\\ 0.0005496\\ \hline $0.0005496\\ 0.0005496\\ 0.0005496\\ \hline $0.0005496\\ 0.0005496\\ 0.0005496\\ \hline $0.0005496\\ 0.0005496\\ \hline $0.0005496\\ 0.0005496\\ \hline $0.0005496\\ 0.0005496\\ \hline $0.0005496\\ \hline $0.0005496\\ 0.0005496\\ \hline $0.0005496\\ \hline $0.000540\\ \hline $0.0005496\\ \hline $0.0005496\\ \hline $0.0005496\\ \hline $0.0005496\\ \hline $0.0005496\\ \hline $0.000540\\ \hline $0.0005$	$\begin{array}{c} \text{-0.4170051} \\ \text{-0.4170051} \\ \text{-0.8127236} \\ \text{-0.3861100} \\ \text{0.0673725} \\ \text{0.0414649} \\ \text{-0.0163873} \\ \hline \\ $	$\begin{array}{c} -5.31 & 1735 \\ 0 & 1795 \\ -586.34 & 15230 \\ 3096.31 & 117752 \\ \hline \\ \mu(k) & \mu_1(k) \\ 1.414 & 0 \\ 0 & 0 & 0.517 \\ 0.189 & 0.189 \\ -0.776 & 0.827 \\ \hline \\ 1.41 & 0 \\ 0 & 0 \\ $	3     6       7     8       8     0       1     2       3     0       1     2       3     4       5     6

Table 1 contains the scaling function and wavelet coeffi-

cients for the length-6 and 12 designed by Daubechies and length-8 designed by Tian together with their discrete moments. We see the extra zero scaling function moments for lengths 6 and 12 and also the extra zero for lengths 8 and 12 that occurs after a non zero one.

The continuous moments can be calculated from the discrete moments and lower order continuous moments [3, 5, 8] by

$$m(k) = \frac{1}{(2^k - 1)\sqrt{2}} \sum_{\ell=1}^k \binom{k}{\ell} \mu(\ell) m(k-\ell) \quad (12)$$

and

$$m_1(k) = \frac{1}{2^k \sqrt{2}} \sum_{\ell=0}^k \binom{k}{\ell} \mu_1(\ell) m(k-\ell).$$
(13)

These equations exactly calculate the moments defined by the integrals in (8) and (3) from simple finite convolutions of the discrete moments with the lower order continuous scaling moments. This allows the recursive calculation of the continuous moments from the discrete moments. Similar equations also hold for the general multiplier-M case.

An important relationship of the discrete moments for a system with K - 1 zero wavelet moments is [5]

$$\sum_{\ell=0}^{k} \binom{k}{\ell} (-1)^{\ell} \mu(\ell) \ \mu(k-\ell) = 0.$$
 (14)

Solving for  $\mu(k)$  in terms of lower order discrete moments and using  $\mu(0) = \sqrt{2}$  gives for k even

$$\mu(k) = \frac{-1}{2\sqrt{2}} \sum_{\ell=1}^{k-1} {\binom{k}{\ell}} (-1)^{\ell} \mu(\ell) \ \mu(k-\ell)$$
(15)

which allows calculating the even-order discrete scaling function moments in terms of the lower odd-order discrete scaling function moments for  $k = 2, 4, \dots, 2K - 2$ . For example:

$$\mu(2) = \frac{1}{\sqrt{2}}\mu^2(1)$$
 (16)

$$\mu(4) = \frac{-1}{2\sqrt{2}} [8\mu(1)\,\mu(3) - 3\mu^4(1)]$$
(17)

which can be seen from values in [9].

Johnson [10] noted from Beylkin [11] and Unser [12] that by using the moments of the autocorrelation function of the scaling function, a relationship of the continuous scaling function moments can be derived in the form

$$\sum_{\ell=0}^{k} \begin{pmatrix} k\\ \ell \end{pmatrix} (-1)^{k-\ell} m(\ell) \ m(k-\ell) = 0 \qquad (18)$$

where 0 < k < 2K if K - 1 wavelet moments are zero. Solving for m(k) in terms of lower order moments gives for k even

$$m(k) = \frac{-1}{2} \sum_{\ell=1}^{k-1} {k \choose \ell} (-1)^{\ell} m(\ell) \ m(k-\ell)$$
(19)

which allows calculating the even-order scaling function moments in terms of the lower odd-order scaling function moments for  $k = 2, 4, \dots, 2K - 2$ . For example [10]:

$$m_2 = m_1^2$$
 (20)

$$m_4 = 4 m_3 m_1 - 3 m_1^4 \tag{21}$$

$$m_6 = 6 m_5 m_1 + 10 m_3^2 + 60 m_3 m_1^3 + 45 m_1^0 (22)$$

$$m_8 = 8 m_7 m_1 + 56 m_5 m_3 - 168 m_5 m_1^3 + 2520 m_3 m_1^5 - 840 m_3 m_1^2 - 1575 m_1^8 (23) \dots \dots \dots$$

if the wavelet moments are zero up to k = K - 1. Notice that setting  $m_1 = m_3 = 0$  causes  $m_2 = m_4 = m_6 = m_8 = 0$ if sufficient wavelet moments are zero. This explains the extra zero moments in Table 1. It also shows that the traditional specification of zero scaling function moments is redundant. In Table 1,  $m_8$  would be zero if more wavelet moments were zero.

# Coiflet Systems from a Specified Filter Length

The preceding section shows that Coifman systems do not necessarily have an equal number of scaling function and wavelet moments equal to zero. Lengths  $N = 6\ell + 2$  have equal numbers of zero scaling function and wavelet moments, but always have even-order "extra" zero scaling function moments located after the first non-zero one. Lengths  $N = 6\ell$  always have an "extra" zero scaling function moment. Indeed, both will have several even-order "extra" zero moments for longer N as a result of the relationships illustrated in (20) through (23). Lengths  $N = 6\ell - 2$  do not occur for the original definition of a Coifman system if one looks only at the degree with minimum length. If we specify the length of the coefficient vector, all even lengths become possible, some with the same coiffet degree.

Table 1 also shows the result of designing a length-4 system, using the one degree of freedom to ask for one zero scaling function moment rather than one zero wavelet moment as we did for the Daubechies system. For length-4, we do not get any "extra" zero moments because there are not enough zero wavelet moments. Here we see a direct trade-off between zero scaling function moments and wavelet moments. Adding these new lengths to our traditional coiflets gives Table 2.

Table 2: Moments for Various Length-N and Degree-L Coiflets, where (\*) is the number of zero wavelet moments, excluding the  $m_1(0) = 0$ 

$\overline{N}$		m = 0	$m_1 = 0$	m = 0	$m_1 - 0$	Zero
1.4		m = 0	$m_1 = 0$	m = 0	$n_1 - 0$	
		set	set.	actual	actuar	mom.
4	1	1	0	1	0	1
6	2	1	1	2	1	3
8	3	2	2	2	2	4
10	3	3	$\overline{2}$	$\overline{4}$	2	6
12	4	3	3	4	3	ž
14	5	4	4	4	4	8
16	5	5	4	6	4	10
18	6	5	5	ő	5	11
20	7	6	6	6	6	12
22	7	7	6	8	6	14
24	8	7	7	8	7	15
26	9	8	8	8	8	16
28	9	9	8	10	8	18
30	10	9	9	10	9	19

The fourth and sixth columns in Table 2 contain the number of zero wavelet moments, excluding the  $m_1(0) = 0$  which is zero because of orthogonality in all of these systems. The extra

zero scaling function moment that occurs just after a non zero moment for  $N = 6\ell + 2$  is also excluded from the count. This table shows coiflets for all even lengths. It shows the extra zero scaling function moments that are sometime achieved and how the total number of zero moments monotonically increases and how the "smoothness" as measured by the Hölder exponent [13] increases with N and L.

## Conclusions

In this paper we have examined the relationship between the number of specified zeros of scaling function moments, wavelet moments, and the length of the scaling filter. By specifying either equal numbers or one more scaling function moment to be zero, all even length scaling filters can be obtained. As noted earlier and illustrated in Table 3, these coiflets fall into three classes. Those with scaling filter lengths of  $N = 6\ell + 2$  (due to Tian) have equal number of zero scaling function and wavelet moments, but always has "extra" zero scaling function moments located after the first non-zero one. Lengths  $N = 6\ell$  (due to Daubechies) always have one more zero scaling function moment than zero wavelet moment and lengths  $N = 6\ell - 2$  (new) always have two more zero scaling function moments than zero wavelet moments. These "extra' zero moments are predicted by (20) - (23) and there are additional even-order zero moments for longer lengths. We have observed that within each of these classes, the Hölder exponent increases monotonically.

Table 3: Number of Zero Moments for The Three Classes of Generalized Coiflets ( $\ell = 1, 2, \cdots$ ), \*excluding  $\mu_1(0) = 0$ , †excluding non-contiguous zeros

N	$m = 0^{\dagger}$	$m_1 = 0^*$	Total zero
Length		achieved	moments
$N = 6\ell + 2$ $N = 6\ell$ $N = 6\ell - 2$	(N-2)/3 N/3 (N+2)/3	$\frac{(N-2)/3}{(N-3)/3}$ $\frac{(N-4)/3}{(N-4)/3}$	$\begin{array}{r} & (2/3)(N-2) \\ (2/3)(N-3/2) \\ (2/3)(N-1) \end{array}$

The approach taken in other investigations of coiflets would specify the coiflet degree and then find the shortest filter that would achieve that degree. The lengths  $N = 6\ell - 2$  were not found by this approach because they have the same coiflet degree as the system just two shorter. However, they achieve two more zero scaling function moments than the shorter length with the same degree. The approach taken in this paper in specifying the number of zero moments and/or the filter length gives more insight and makes it easier to see the complete picture.

In addition to the variety of coiflets obtained by changing the moment and/or the length specifications, many (perhaps all) of these sets of specified zero moments have multiple solutions. This is certainly true for length-6 as illustrated in (10) through (11) and for other lengths that we have found experimentally. The variety of solutions for each length can have different shifts, different Hölder exponents, and different degrees of being approximately symmetric.

It may be that setting a few scaling function moments and a few wavelets moments to zero may be sufficient with the remaining degrees of freedom used for some other optimization, either in the frequency domain or in the time domain. An alternative might be to minimize a larger number of various moments rather than to zero a few [14].

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