FREQUENCY SAMPLING FILTERS WITH ALGEBRAIC INTEGERS

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ABSTRACT

Algebraic integers have been proven beneficial to DFT and non-recursive FIR filter designs [2, 4] since algebraic integers can be dense in $\mathbb{C}$, resulting in short word width, high speed designs. This paper uses another property of algebraic integers: algebraic integers can produce exact pole zero cancellation pairs that are used in recursive FIR, frequency sampling filter designs.

1. INTRODUCTION

An element of $\mathbb{C}$ is an algebraic integer if it is a zero of a monic polynomial in $F[x]$ where F is one of the fields $\mathbb{Z}, \mathbb{Z}_M$, or $\mathbb{Z}_p$ [1, p. 269]. M is taken to be composite and p is assumed to be prime. If R is a commutative ring with unity and $p(x)$ is an irreducible polynomial in $R[x]$ then the quotient ring $R[x]/\langle p(x)\rangle$ is a field. Therefore, for non-zero N in such a field, N^{-1} exists. For frequency sampling filters (FSFs), if $p(x)$ is monic then the ring properties of $\mathbb{Z}_M/\langle p(x)\rangle$ are sufficient and used in the remainder of the paper. Now, $p(x)$ will be selected so that algebraic integers can be used to describe the complex plane.

If $p(x) = x^N - 1$ then $\mathbb{Z}_M[x]/\langle x^N - 1\rangle \cong \mathbb{Z}_M[W_N]$ where $W_N = e^{j2\pi/N}$. The quotient ring $\mathbb{Z}_M[x]/\langle x^N - 1\rangle$ is cyclic and has order N. Addition of polynomials $A(x), B(x) \in \mathbb{Z}_M[x]/\langle x^N - 1\rangle$ is given by

$$A(x) + B(x) = \sum_{k=0}^{N-1} (a_k + b_k)\, x^k, \qquad (1)$$

and multiplication is given by

$$A(x) \cdot B(x) = \sum_{k=0}^{N-1} \left(\sum_{l=0}^{N-1} a_l b_{\langle k-l\rangle_N} \right) x^k, \qquad (2)$$

where $\langle\cdot\rangle_m$ is modular reduction of $\cdot$ modulo m. Addition of polynomials in the field $\mathbb{Z}_M[x]/\langle x^N - 1\rangle$ is the usual component-wise addition operation. However, the multiplication of polynomials given above is recognized as cyclic convolution. An interesting property of the multiplication given above is that if $B(x) = x^l$ then the product is simply a cyclic rotation of the coefficients of $A(x)$.

Cozzens and Finkenstein [2] have shown that the quotient ring of algebraic integers produce for $N \geq 8$ a dense set in the complex plane. The benefit for a DFT implementation is that greater accuracy may be achieved using fewer bits than necessary with a conventional approach. A potential decrease in system complexity has also been suggested [2]. Not all N components $e_k = W_N^k$ are necessary; most of them are linear combinations of the others. Only those components e_k must be kept that are totative to N, that is, $\gcd(N, k) = 1$. Instead of $x^N - 1$, it is sufficient to use the *cyclotomic polynomial* [12, 9]

$$p(x) = C_N(x) = \prod_{\substack{\gcd(N,k)=1 \\ 0<k<N}} x - W_N^k. \qquad (3)$$

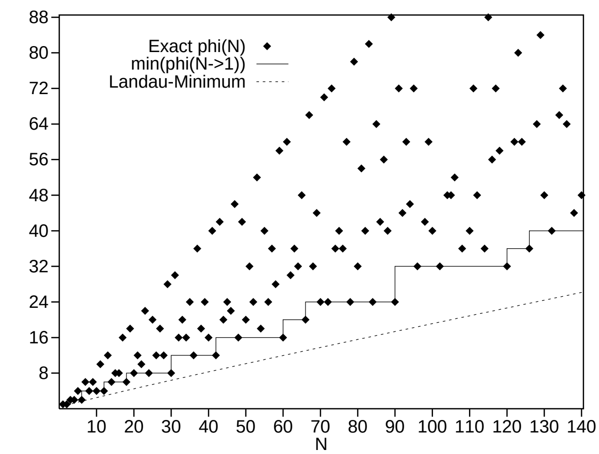

Figure 1: Order of $C_N(x)$, which is $\phi(N)$ (symbol ⧫). Computation of lower bound (solid line) starting with a high N value (1000) and computation of $\phi(N)$ down to $N = 1$. Lower bound approximation by Landau (dashed line).

Cozzens and Finkenstein [2] suggest the use of a power of two length, $N = 2^l$, for the DFT computation ($\phi(2^l) = 2^{l-1}$). In FSF designs, the aim is to generate as many points N as possible on the unit circle for a fixed $\phi(N)$. A lower bound for $\phi(N)$ is provided by Landau [7],

$$\frac{\phi(N)}{N} \quad \geq \quad \frac{e^{-\gamma}}{\log_e \log_e(N) + 3.51}, \qquad (4)$$

where γ is the Euler constant ($\gamma \approx 0.5772156649$). From Figure 1 it can be seen that the Landau bound is not accurate enough for small values of N. The following table provide the optimal choice of N (i.e., maximal N for fixed $\phi(N)$).

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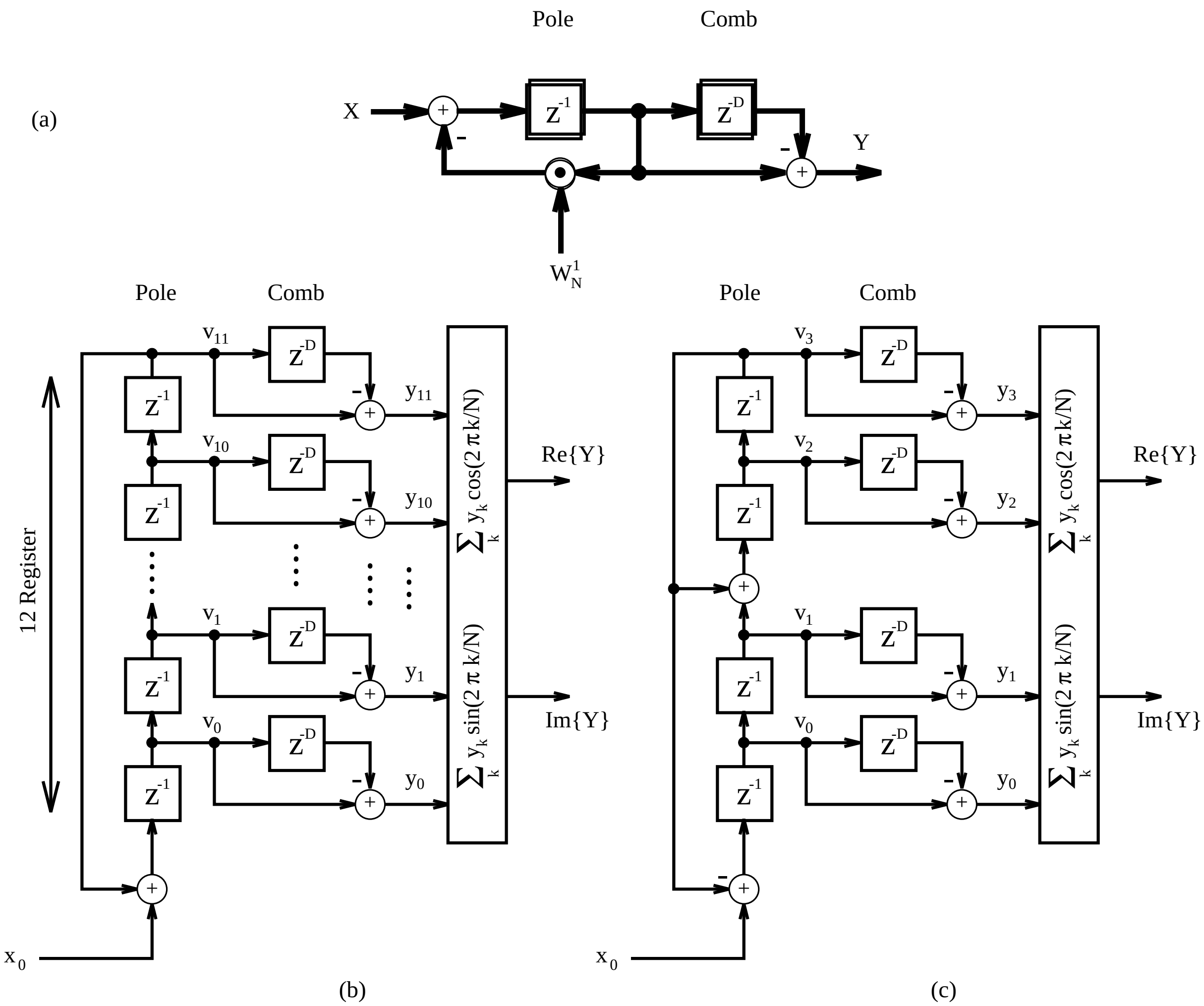

Figure 2: One stage FSF for $N = 12$ and pole at W_{12}^1. (a) One pole FSF filter (shadowing symbolizes the processing in parallel channels for algebraic integers). (b) Without cyclotomic polynomial reduction. (c) Using the cyclotomic polynomial $C_{12}(x)$.

$\phi(N)$	1	2	4	6	8	12
N	2	6	12	18	30	42
$\phi(N)$	16	20	24	32	36	40
N	60	66	90	120	126	150

For example, choose $\phi(N) = 2$ as the number of components per algebraic integer. Then $N = 6$ exact roots on the unit circle can be designed using $e_0 \cong 1$, $e_1 \cong x$, and the cyclotomic polynomial, $C_6(x) = x^2 + x + 1$. The remaining algebraic integers may be constructed using $e_2 = e_1 - e_0 \cong x - 1$, $e_3 = -e_0 \cong -1$, $e_4 = -e_1 \cong -x$, and $e_5 = e_0 - e_1 \cong 1 - x$.

2. FREQUENCY SAMPLING FILTERS

A classical FSF consists of a comb filter cascaded with a bank of frequency selective resonators [15]. The resonators independently produce a collection of poles that annihilate the zeros produced by the comb pre-filter. Gain adjustments are applied to the output of the resonators so as to approximately profile the magnitude frequency response of a desired filter. An FSF can also be created by cascading all-pole filter sections with all-zero filter (comb) sections as suggested in Figure 2(a). The delay of the comb section $1 \pm z^{-D}$ is chosen so that its zeros cancel the poles of the all-pole pre-filter. It can be observed that wherever there is a complex pole, there also exists an annihilating complex zero. The filter has pole/zero symmetry to the unit cycle which results in linear phase and constant group delay properties. FSFs of this type are known to provide very efficient multi-rate interpolation and decimation solutions and may serve as high decimation rate filters for RF-to-baseband conversion of radio signals [5, 6]. If the filter of Figure 2(a) is realized with a non-recursive FIR, then D (complex) multiplications and $D - 1$ additions are used. In contrast, the recursive design uses only one multiplication and one subtraction!

2.1. Improvement of Frequency Selective Properties Using Algebraic Integers

To motivate this discussion, consider again the filter shown in Figure 2(a). It can be argued that first-order filter sections produce poles at angles 0° and 180°. Second-order pole sections with

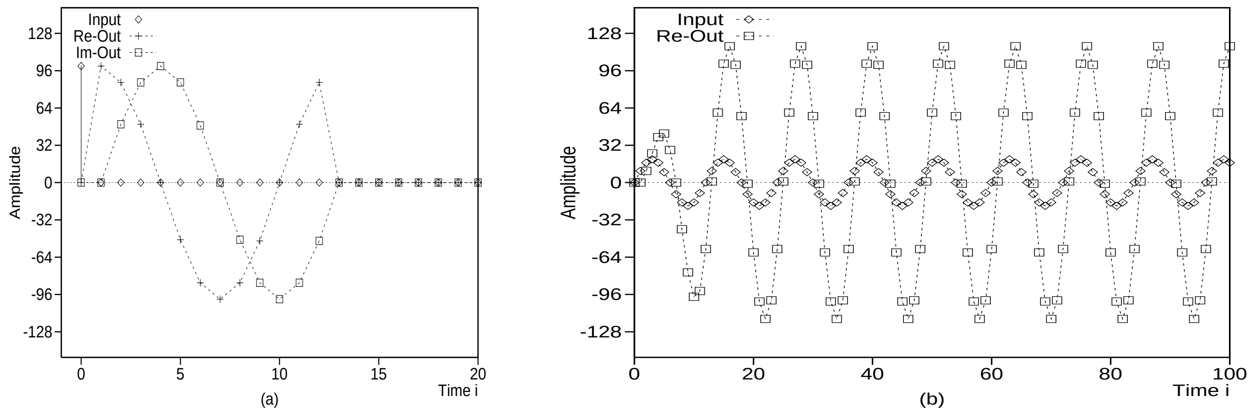

Figure 3: (a) Impulse response for the filter from Figure 2(a) with $D = 12$. (b) Eigenfrequency test $D = 12$.

integer coefficients can produce poles at angles 60°, 90°, 120° according to the relationship $2\cos(2\pi K/D)$=1, 0, and -1. For sections of higher order, only multi-passband filters can be implemented with integer coefficients [10]. The design algorithm previously proposed by the authors [10] works well for filter banks with few channels. This FSF design paradigm produces poor results for filter banks with fifteen to twenty channels, such as those used in high quality speech processing. This is because higher order pole sections generate multiple passbands and the complexity for the anti-aliasing filter is greater than that of the FSF section.

Now, the algebraic integers introduced in the first section are used to construct *single* passband building blocks. The conversion from complex numbers to algebraic integers (with $N > 4$) has been investigated [3, 8]. A direct approach for conversion of real integers to algebraic integers uses only the first component, forcing all other components to zero (i.e., $A = (a_0, 0, 0, \ldots)$). The conversion from algebraic integers to complex numbers will be shown to be efficiently implemented using the CORDIC algorithm [14, 11], where

$$\begin{aligned} \text{Re}\{A\} &= \sum_k a_k \cos(2\pi k/N) \\ \text{Im}\{A\} &= \sum_k a_k \sin(2\pi k/N). \end{aligned}$$

2.2. Implementation Issues of Algebraic Integer Filters

Cozzens, Finkelstein, and Games [2, 3] have defined algebraic numbers over the cyclotomic polynomial $C_N(x)$, (i.e., the quotient ring $\mathbb{Z}_M/\langle C_N(x)\rangle$) to lower the number of vectors from N to $\phi(N)$, where $\phi(N)$ is the Euler totient function. A polynomial $A(x) \in \mathbb{Z}_M/\langle C_N(x)\rangle$ is expressed as

$$A(x) = a_0 + a_1 \cdot x + a_2 \cdot x^2 + \ldots + a_{\phi(N)-1} \cdot x_N^{\phi(N)-1},$$

with $a_i \in \mathbb{Z}_M$. Addition of two polynomials, $A(x)$ and $B(x)$, in this ring is component-wise and is given by

$$A(x) + B(x) = \sum_{k=0}^{\phi(N)-1} (u_k + v_k)\ x^k. \tag{5}$$

The multiplication is a convolution sum of the coefficients modulo the cyclotomic polynomials $C_N(x)$. That is,

$$\begin{aligned} A(x) \cdot B(x) &= (a_0 b_0) + (a_0 b_1 + a_1 b_0)\ x \\ &\quad + (a_0 b_2 + a_1 b_1 + a_2 b_0)\ x^2 + \ldots. \end{aligned} \tag{6}$$

Because algebraic integers can be used to represent N exact numbers located on the unit circle, cyclotomic polynomials can provide a framework for pole assignment of a single frequency filter having poles on the unit circle.

2.3. Example Designs for Algebraic Integer Processing

Figure 2(a) presents a single stage, single passband FSF with algebraic integer processing, having a *single* pole angle of $360°/12 = 30°$. Figures 2(b) and 2(c) shows two realization options, Figure 2(b) without cyclotomic polynomial reduction (i.e., $p(x) = x^{12} - 1$), and Figure 2(c) uses algebraic integers over the cyclotomic polynomial $C_{12}(x) = x^4 - x^2 + 1$. For the realization found in Figure 2(b) twelve components are used, resulting in twelve comb sections. The realization with the cyclotomic polynomial requires one more subtraction for the pole realization, but only four comb sections. The complexity reduction through the use of cyclotomic polynomials is obvious.

To simplify the graphical representation, the simulations are performed with symmetric, (two's Complement) *single* modulus RNS arithmetic. Nevertheless, a high speed implementation should use the multi-modulus RNS arithmetic isomorphism

$$\mathbb{Z}_M \cong \mathbb{Z}_{m_1} \times \mathbb{Z}_{m_2} \times \cdots \times \mathbb{Z}_{m_L} \tag{7}$$

within a set of relatively-prime, independent, small word length channels m_k [13]. Figure 3(a) shows the impulse response of the filter from Figure 2(a) with a comb delay $D = 12$. Figure 3(b) shows the result of an eigenfrequency test with an input signal $x[i] = (a_0[i], 0, \ldots) = (20\sin(2\pi i/12), 0, \ldots)$. Both realizations are shown in Figures 2(b) and 2(c). Neglecting the quantization error from sine and cosine functions, the architectures generate identical impulse responses and eigenfrequency test results. From Figure 4, it is obvious that the realization with cyclotomic polynomial reduction gives higher incidence of modular over- and underflows (indicated with the arrows ↑↓). It can also be seen by comparing Figure 3 and Figure 4 that all overflows do not

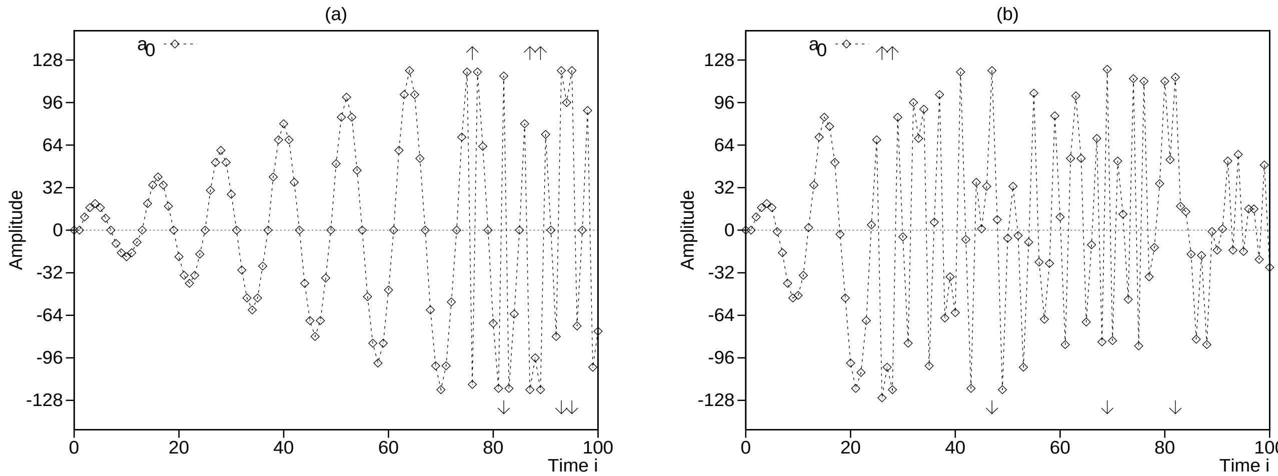

Figure 4: Overflow behavior to Figure 3 for v_0 with eigenfrequency test $D = 12$. Modulus is 256. (a) Pole without modulo reduction. (b) Pole with modulo reduction.

result in unacceptable behavior since the filters are implemented using exact $\mathbb{Z}_M$ arithmetic. The filter behaves identically to a non-recursive FIR realization. To compare this realization with that previously found by the authors [10], it should be emphasized that the algebraic integer realization has higher complexity (four times as many comb sections) but has a single complex pole and is therefore a *single* passband filter and consequently does *not* need an additional anti-aliasing filter.

3. CONCLUSION

The Hogenauer [6] idea of cascade integrator comb filter was extended to bandpass-filters. Using a digital signal processing scheme with algebraic integers provides single passband frequency sampling filter building blocks. These filters are of low complexity and are multiplier free, so that a wide selection of passband frequencies may be implemented without the high cost of anti-aliasing filters as previously proposed [10].

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