

# MULTIPLE WINDOW NON-LINEAR TIME-VARYING SPECTRAL ANALYSIS

*Ferhat Çakrak and Patrick J. Loughlin*

Dept. of Electrical Engineering  
University of Pittsburgh  
Pittsburgh, PA 15261, USA  
ferhat@ee.pitt.edu, pat@ee.pitt.edu

## ABSTRACT

A non-linear multi-window method for generating a time-varying spectrum of non-stationary signals in noise is presented. The time-varying spectrum is computed from an optimally weighted average of multiple Hermite windowed spectrograms. The weights are determined using linear least squares estimation with respect to a reference time-frequency distribution. A masking operation is also used to reduce extraneous side lobes introduced by higher order Hermite windows. Several examples are provided, with performance criteria measures, to demonstrate and quantify the effectiveness of this new method.

## 1. INTRODUCTION

Time-varying spectral analysis is important in many applications such as radar, sonar, speech, geophysics and biological signals. Many methods have been developed, particularly for deterministic signal analysis, and perhaps the most popular method is the spectrogram, which is the magnitude squared of the short-time Fourier transform [1], [2], [11]. For stochastic processes, one of the primary methods for time-frequency analysis is the Wigner-Ville spectrum (WVS), defined by Martin and Flandrin as [12]

$$\mathbf{W}(t, f) = E \left\{ \int x^* \left( t - \frac{\tau}{2} \right) x \left( t + \frac{\tau}{2} \right) e^{-j2\pi f \tau} d\tau \right\} \quad (1)$$

where the integral is the Wigner distribution (WD) of the process  $x(t)$ , and  $E\{\cdot\}$  denotes the expectation operator.

If several realizations of the nonstationary process  $x(t)$  are available, one can obtain an estimate of the WVS by ensemble averaging the individual WD's of each realization. Often, however, only a single realization of the process is available, in which case some smoothing of the WD of  $x(t)$  is typically performed to estimate the WVS [12]. Smoothing with a Born-Jordan kernel yields a minimum variance estimate of the WVS of a white noise process [3], [9]. If the statistics (up to fourth order) are known for a signal + noise process, then a non-trivial signal-dependent kernel can be derived to optimally smooth the WD and estimate the WVS [14]. If these statistics are not known, effective smoothing can still be achieved, utilizing separable kernels as in [12], or signal-dependent methods as in [10].

Recently, an alternative approach to WVS estimation was developed by Bayram and Baraniuk [5], [6]. They extended Thomson's multiple window stationary spectral estimation procedure [16] to the nonstationary case by averaging multiple Hermite

windowed spectrograms of the de-chirped process  $x(t)$ . The de-chirping procedure removes dominant line components of  $x(t)$  in the time-frequency plane, and therefore a priori knowledge of the signal is required. While effective, their method has limitations, as they note [5], [6]. For example, the chirp extraction procedure is computationally very demanding, and fails when two or more crossing chirps are present in the signal.

In this paper, we continue the development of methods for nonstationary WVS estimation, and in particular the multi-window approach initiated by Bayram and Baraniuk. We develop a non-linear least-squares multi-window time-varying spectral estimation procedure that addresses some of the limitations noted by Bayram and Baraniuk. The method we develop does not require extraction of chirp components, works well with multiple component signals (including crossing chirps), and is computationally simple. Examples are provided, with performance criteria measures, to demonstrate and quantify the effectiveness of this new method.

## 2. MULTIPLE WINDOW NON-LINEAR TIME-VARYING SPECTRUM

Given a realization of a discrete complex time-varying noise corrupted process,

$$x(n) = s(n) + \eta(n) \quad (2)$$

where  $s(n)$  and  $\eta(n)$  correspond to signal and noise components respectively, our aim is to generate a computationally efficient, high resolution time-varying spectral estimate of the noisy signal with good performance in low signal to noise ratio (SNR). We achieve this goal by computing a non-linear multi-window weighted average combination of spectrograms closest to the WD in a least squares sense.

Given  $x(n)$  we calculate  $K$  ( $k = 0, 1, \dots, K - 1$ ) different Hermite windowed spectrograms,

$$S_k(n, m) = \left| \sum_{\ell=-L/2}^{L/2-1} x(n + \ell) h_k(\ell) e^{-j2\pi \ell m/L} \right|^2 \quad (3)$$

where the Hermite windows,

$$h_k(t) = \frac{1}{\sqrt{\sqrt{\pi} 2^k k!}} (-1)^k e^{\frac{t^2}{2}} \left( \frac{d}{dt} \right)^k e^{-t^2} \quad (4)$$

are optimally concentrated in the circular time-frequency region  $L_R = \{(t, f) : t^2 + f^2 \leq R^2\}$  and form an orthonormal basis in  $L^2(\mathbb{R})$  [8]. Then we optimally weight each of the spectrograms

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by solving the linear LS problem

$$\min_{d_k} \sum_n \sum_m \left| P(n, m) - \sum_{k=0}^{K-1} d_k S_k(n, m) \right|^2 \quad (5)$$

where  $P(n, m)$  is a reference time-frequency distribution, which we take here to be the *WD*.

Using a one dimensional indexing (via row-wise scanning) of  $S_k(n, m)$  and  $P(n, m)$ , (5) can be written compactly in matrix form as

$$\min_d \|\mathbf{P} - \mathbf{S}\mathbf{d}\|^2. \quad (6)$$

The linear LS solution of (6) is given by

$$\mathbf{d}^{\text{opt}} = (\mathbf{S}^T \mathbf{S})^{-1} \mathbf{S}^T \mathbf{P}. \quad (7)$$

A linear least-squares estimate  $\hat{P}_L$  of  $\mathbf{P}$  is

$$\hat{P}_L(n, m) = \sum_{k=0}^{K-1} d_k^{\text{opt}} S_k(n, m). \quad (8)$$

Since negative values have no physical interpretation in a time-varying spectrum, we take the positive part of (8). Furthermore, since the  $k$ -th order Hermite window  $h_k(\ell)$  has  $k$  zeros which show up as side-lobes in the multiple window spectrum, we mask the positive part of the spectrum in order to eliminate these side-lobes as follows. Noting that the zero associated with  $h_1(\ell)$  is hidden under the main lobe associated with  $h_0(\ell)$ , we observe that the positive part of the estimate obtained for  $K = 2$  will have no side-lobes. Therefore, we can use the multiple window spectrum (*MWS*) for  $K = 2$  to mask the positive part of the spectrum for  $K > 2$ . The masking function  $\Gamma(n, m)$  is

$$\Gamma(n, m) = \begin{cases} 1 & \text{if } \hat{P}_2(n, m) \geq 0 \\ 0 & \text{if } \hat{P}_2(n, m) < 0 \end{cases}. \quad (9)$$

Our non-linear multiple window time-varying spectrum is thus

$$\hat{P}(n, m) = \Gamma(n, m) \left[ \left( \sum_{k=0}^{K-1} d_k^{\text{opt}} S_k(n, m) \right)^+ \right] \frac{E}{\|\hat{P}(n, m)\|}$$

where  $\Gamma(n, m)$  and  $(\cdot)^+$  denote the masking and positive-only thresholding operations respectively, and the scalar  $\frac{E}{\|\hat{P}(n, m)\|}$  normalizes the distribution such that its total energy equals the total energy ( $E$ ) of the given signal.

The main computational requirement of our method is the calculation of optimal weightings for the Hermite windowed spectrograms. This requires the inversion of the  $K \times K$  matrix  $\mathbf{B} = \mathbf{S}^T \mathbf{S}$  which can easily be carried out by using the Cholesky factorization technique, due to the fact that  $\mathbf{B}$  is a symmetric positive definite matrix.

### 3. EXAMPLES

In this section, we give several examples for noise-free and noisy signals. The effectiveness of the non-linear *MWS* is quantified via a performance measure, with comparisons to other techniques, and to an “ideal” *TFR* which we use as the standard of comparison, generated as follows.

It is well known that, given only the mean  $m_x$  and variance  $\sigma_x^2$  of an unknown density  $f(x)$ , the maximum entropy estimate is

Gaussian,  $N(m_x, \sigma_x^2)$  [13], [15]. Furthermore, given  $L$  independent (therefore uncorrelated) random variables  $x_i$  with means  $m_i$  and variances  $\sigma_i^2$ ,  $i = 1, 2, \dots, L$ , the maximum entropy density is a sum of Gaussian densities [13], [15],

$$f(\mathbf{x}) = \frac{1}{L} \sum_{i=1}^L \frac{1}{\sqrt{2\pi\sigma_i^2}} e^{-(x_i - m_i)^2 / 2\sigma_i^2}. \quad (10)$$

In time-frequency theory, the mean frequency of a signal component at a given time is given by the instantaneous frequency,  $\varphi'_i(t)$ , of the component [7]. Similarly, the variance in frequency at a given time is given by the (square of) instantaneous bandwidth,  $\sigma_i^2(t)$  [7]. Hence, given the (statistically independent) individual instantaneous frequencies and bandwidths of a multi-component signal, it follows from above that the maximum entropy spectral density at a given time is

$$\bar{P}(\omega|t) = \sum_{i=1}^L \frac{p_i}{p} \frac{1}{\sqrt{2\pi\sigma_i^2(t)}} e^{-(\omega - \varphi'_i(t))^2 / 2\sigma_i^2(t)} \quad (11)$$

where  $\bar{P}(\omega|t)$  denotes a conditional density, and  $\frac{p_i}{p}$  is a normalization factor which is the ratio of the power of the  $i^{\text{th}}$  signal component to the total signal power over the time-frequency interval considered. The maximum entropy joint density (*MED*) is therefore given by

$$\begin{aligned} \bar{P}(t, \omega) &= \bar{P}(t) \bar{P}(\omega|t) \\ &= \sum_{i=1}^L \frac{p_i}{p} \frac{1}{\sqrt{2\pi\sigma_i^2(t)}} \bar{P}_i(t) e^{-(\omega - \varphi'_i(t))^2 / 2\sigma_i^2(t)} \end{aligned} \quad (12)$$

where  $\bar{P}_i(t)$  is the time marginal of the  $i^{\text{th}}$  signal component.

We use this *MED* to achieve an objective comparison between *TFRs* for known signals. We define a performance measure (*PM*) of the resulting *TFRs* with respect to this ideal as

$$PM = 10 \log_{10} \left( \frac{\sum_{n=1}^N \sum_{m=1}^M \bar{P}^2(n, m)}{\sum_{n=1}^N \sum_{m=1}^M |\bar{P}(n, m) - \hat{P}(n, m)|^2} \right) \text{ (dB)}$$

where  $\bar{P}(n, m)$  is the discrete *MED* of the noise-free signal, and  $\hat{P}(n, m)$  is the *TFR* being assessed.

For comparison purposes, we generated several discrete synthetic test signals of the form

$$x(n) = s(n) + \eta(n) = \sum_{i=1}^L \mu_i(n) e^{j\varphi_i(n)} + \eta(n) \quad (13)$$

where

$$\mu_i(n) = c_i (\alpha_i / \pi)^{1/4} e^{-\alpha_i (n - n_{0i})^2 / 2} \quad (14)$$

and

$$\varphi_i(n) = \gamma_i n^3 / 3 + \beta_i n^2 / 2 + \tilde{\omega}_{0i} n \quad (15)$$

are the time-varying amplitude and phase, respectively of the  $i^{\text{th}}$  signal component,  $\eta(n)$  is complex additive Gaussian white noise (*AGWN*),  $\tilde{\omega}_{0i}$  is the discrete-time frequency in radians, and  $c_i = (\frac{p_i}{p})^{1/2}$ . From (14) and (15), we have  $\varphi'_i(n) = \gamma_i n^2 + \beta_i n + \tilde{\omega}_{0i}$  and  $\sigma_i^2(n) = \frac{1}{2\alpha_i}$ , from which one can easily construct the *MED* of the noise-free signal, per (12). We define the *SNR* of a noise corrupted signal as

$$SNR = 10 \log_{10} \left( \frac{\frac{1}{N} \sum_{n=1}^N |s(n)|^2}{\frac{1}{N} \sum_{n=1}^N |\eta(n)|^2} \right) \text{ (dB)}. \quad (16)$$

*Example 1:* Fig. 1 shows six time-frequency representations of three discrete complex Gaussian parallel chirp signals where  $c_1 = c_2 = c_3 = 1$ ,  $\beta_1 = \beta_2 = \beta_3 = \pi/N$ ,  $\alpha_1 = \alpha_2 = \alpha_3 = 0.006$ ,  $n_{01} = 32$ ,  $n_{02} = 96$ ,  $n_{03} = 160$ ,  $\tilde{\omega}_{01} = \tilde{\omega}_{02} = \tilde{\omega}_{03} = 0$ , and the signal length  $N = 192$ . The  $SNR$  of the signal is 3 dB. Fig. 1(a)-(b) show the *MED* and the proposed non-linear *MWS* for the noise free case, respectively. The well-known cross-terms problem of the *WD* is seen in Fig. 1(c) for the noisy signal. In fact one of the cross terms directly interferes with the chirp component in the middle. The radially Gaussian kernel (*RGK TFR*) [4] with kernel volume  $\alpha = 3$  is shown in Fig. 1(d). The spectrogram with a Gaussian window is shown in Fig. 3(e), and the proposed non-linear *MWS* is shown in Fig. 1(f). The *PM* results for this example are summarized in table 1. As seen from the figure and table, the proposed non-linear *MWS* performs well.

*Example 2:* In this example we consider a multicomponent signal whose components intersect. Fig. 2(a) shows the proposed non-linear *MWS* of two quadratic discrete complex Gaussian crossing chirp signals where  $c_1 = c_2 = c_3 = 1$ ,  $\gamma_1 = -\gamma_2 = \pi/N^2$ ,  $\beta_1 = \beta_2 = 0$ ,  $\alpha_1 = \alpha_2 = 0.0005$ ,  $n_{01} = n_{02} = 128$ ,  $\tilde{\omega}_{01} = 0$ ,  $\tilde{\omega}_{02} = \pi$ , signal length  $N = 256$ , and  $K = 6$ . Performance measures for various  $SNR$  levels are given in Fig. 2(b).

*Example 3:* Fig. 3 shows six time-frequency representations of a multicomponent signal consisting of a tone, linear chirp, and sinusoidal FM. Parameters are  $c_1 = 0.81$ ,  $c_2 = 1$ ,  $c_3 = 0.9$ ,  $\alpha_1 = \alpha_2 = 20/N^2$ ,  $\alpha_3 = 40/N^2$ ,  $n_{01} = n_{02} = 128$ ,  $n_{03} = 102$ ,  $\beta_1 = \beta_2 = 0$ ,  $\beta_3 = (96/N^2)\pi$ ,  $\tilde{\omega}_{01} = (60/N)\pi$ ,  $\tilde{\omega}_{02} = (220/N)\pi$ ,  $\tilde{\omega}_{03} = (120/N)\pi$ , and the signal length  $N = 256$ . In addition, the first signal component has an additional sinusoidal phase of  $20\cos(2\pi n/N)$ . Fig. 3(a)-(b) show the *MED* and the proposed non-linear *MWS* for the noise free case, respectively. Fig. 3(c)-(f) show the *WD*, *RGK TFR*, spectrogram, and non-linear *MWS* in presence of zero mean AGWN with  $SNR = 3$  dB. The *PM* results for this example are summarized in table 1. Here again, the proposed non-linear *MWS* performs well.

*Example 4:* Fig. 4 shows four *TFRs* of the echo-location pulse emitted by the Large Brown Bat, *Eptesicus Fuscus*. The duration of the digitized data is 2.5 microseconds, and there are 400 samples. The sampling period was 7 microseconds. Comparing the non-linear *MWS* to three other *TFRs* in Fig. 4, we see that proposed method is able to detect even the weakest high frequency line component successfully.

#### 4. CONCLUSIONS

In this paper, we developed a simple, effective multi-window method for time-frequency analysis of nonstationary processes. The proposed method was evaluated using clean and noisy signals and compared to the *WD*, spectrogram and other *TFRs*. Quantitative performance measures were given, and in all cases the proposed technique outperformed other methods.

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Table 1: *PM* Results in dB for Example 1 (Fig. 1) and Example 3 (Fig. 3) in noise free and noisy ( $SNR = 3$  dB) cases.

METHOD	Example 1		Example 3	
	Noise Free	Noisy	Noise Free	Noisy
<i>MWS</i>	11.6	6.4	9.0	5.9
<i>RGK TFR</i>	7.8	4.8	6.9	3.9
<i>SGRAM</i>	2.9	2.4	2.2	2.0
<i>WD</i>	-3.0	-3.9	-2.8	-3.4

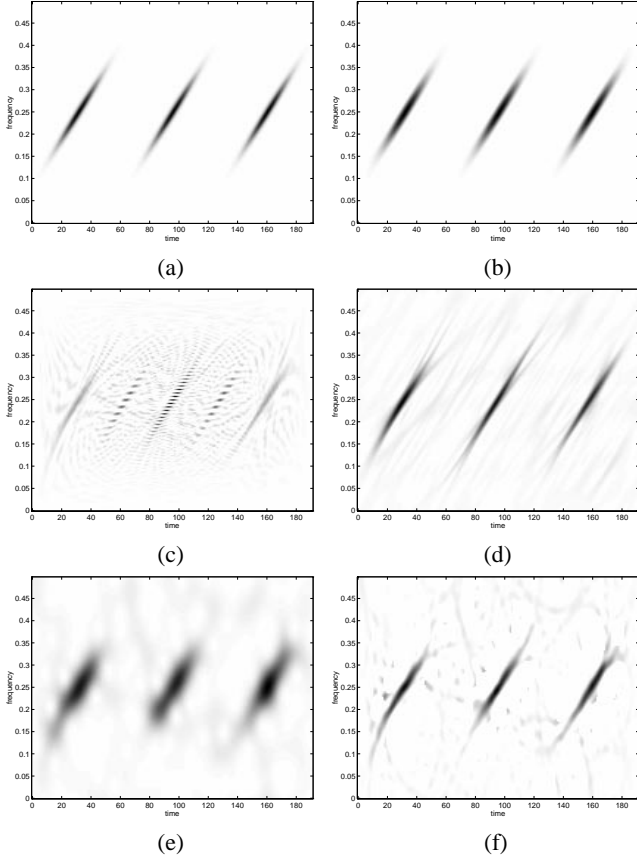


Figure 1: (a) Ideal *MED* of three parallel noise-free Gaussian chirps. (b) Proposed non-linear *MWS* with  $K = 6$  for noise-free case. Four *TFRs* in presence of *AGWN* with  $SNR = 3$  dB: (c) *WD*, (d) *RGK TFR* with kernel volume  $\alpha = 3$ , (e) spectrogram, and (f) proposed non-linear *MWS* with  $K = 6$ . Table 1 gives quantitative performance measures.

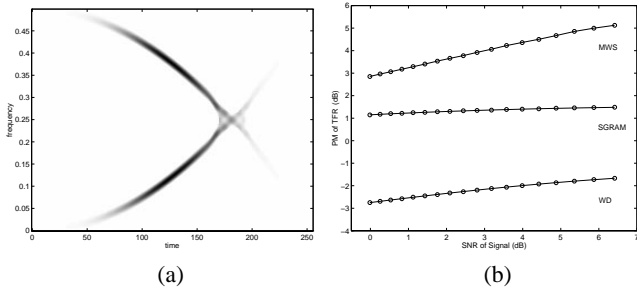


Figure 2: (a) Proposed non-linear *MWS* of two crossing quadratic Gaussian chirps. (b) PM comparison of three *TFRs* of two crossing quadratic chirps in various levels of noise. *TFRs* are the Wigner distribution (*WD*), spectrogram (*SGRAM*), and the proposed non-linear multiple window spectrum (*MWS*).

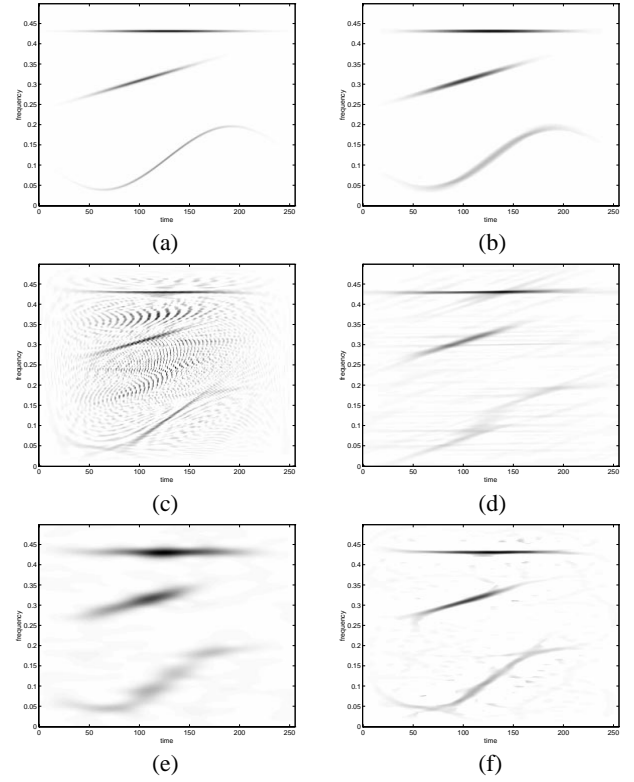


Figure 3: (a) Ideal *MED*, (b) Proposed non-linear *MWS* with  $K = 6$  of a multicomponent signal in noise-free case. Four time-frequency representations of the same multicomponent signal in *AGWN* with  $SNR = 3$  dB: (c) *WD*, (d) *RGK TFR* with kernel volume  $\alpha = 3$ , (e) spectrogram with a Gaussian window, (f) proposed non-linear *MWS* with  $K = 6$ . Table 1 gives quantitative performance measures.

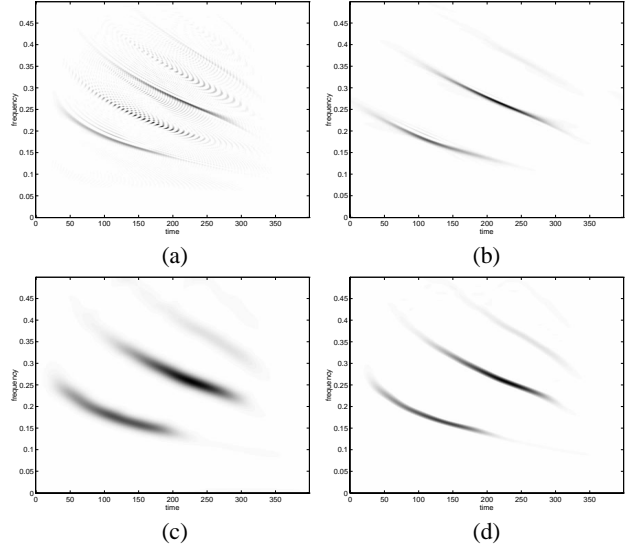


Figure 4: Four *TFRs* of the echo-location pulse emitted by the Large Brown Bat, *Eptesicus Fuscus*: (a) *WD*, (b) *RGK TFR* with kernel volume  $\alpha = 3$ , (c) spectrogram with a Gaussian window, (d) proposed non-linear *MWS* with  $K = 4$ .