CONTINUOUS-TIME RECONSTRUCTION OF NONUNIFORMLY SAMPLED SIGNALS ON A BAND-LIMITED WAVELET BASIS

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ABSTRACT

We propose a reconstruction method of continuous-time random signals by fitting nonuniform samples to a band-limited continuous-time wavelet basis. Based on wavelet analysis, our method uses a windowing technique with variable-sized intervals, taking advantage of the nonuniform signal sampling. This method leads to analytical formulas for the reconstructed continuous-time signal, and as well as for its derivatives. This can be very useful to perform a parametric estimation of so-called continuous-time ARMA models adopted for continuous-time random signals modeling. Several parameters like mother wavelet type, time shift interval between consecutive wavelets and resolution levels number can be adapted, function of nature of nonuniformly sampled signal. In this paper, we describe the principle of the proposed reconstruction method and discuss its performances.

1. INTRODUCTION

One of the usual tasks in digital signal processing is the spectral analysis of sampled signals. Traditionally, the power spectral density (PSD) of a band-limited signal is estimated by applying the discrete Fourier transform (DFT) or the fast Fourier transform (FFT) to a samples sequence obtained by uniform sampling (i.e., at periodic instants) of a continuous-time process. However, in the analysis of real-world measurements, nonuniformly sampled processes occur in various applications. In some of these applications (laser velocimetry, Doppler odometry, microscopic particles counting, radioactivity measurements, radar signal processing etc.) data are nonuniformly time spaced with no underlying basic sampling interval [5]. In some others applications (speech or video transmission on ATM networks, signals transmission etc.) or in the case of temporary malfunctions of measurement sensors, the nonuniform sampling may be due to missing observations. In this last case the observations can occur only at multiples of some « hidden » sampling period [4].

Traditional signal processing techniques are irrelevant for nonuniformly sampled signals. A possible solution is to reconstruct the original continuous-time signal by interpolating the nonuniform samples before any other analysis step. We propose an original method to perform the interpolation of nonuniformly sampled random signals via a band-limited continuous-time wavelet basis fitting. Using a windowing technique with variable-sized intervals allows the proposed method to take advantage of the nonuniform signal sampling. Indeed, wavelet analysis allows the use of long time intervals where low frequency information is relevant, and shorter regions to extract high frequency information. A « Multiresolutional Orthogonal Basis Interpolation (MOBI) » was proposed in [2], only in the case of missing observations, and involved a discrete wavelet transformation (DWT).

We are interested in the most general case of nonuniform sampling where the observations occur at any positive real sampling instants. Our method gives an analytical formula of the reconstructed continuous-time signal. Consequently, the analytical formulas of its derivatives are established. Analog linear filtering of white Gaussian noise being the classical model adopted to model continuous-time random signal [5], a parametric estimation of the continuous-time ARMA model can be performed by using analytical formulas of the reconstructed signal and its derivatives.

Section 2 describes the principle of the reconstruction method via band-limited continuous-time wavelet basis as well as discusses the advantages of this method and its computational requirements. Section 3 presents experimental results and analyses performances of the algorithm. Finally, Section 4 summarizes our work and outline future developments.

2. RECONSTRUCTION METHOD VIA BAND-LIMITED WAVELET BASIS FITTING

2.1. Wavelet series transform

Wavelet series (WS) have been introduced under the form of a signal decomposition on a basis of continuous-time orthogonal wavelets [6].

WS coefficients are defined as sampled continuous wavelet transform (CWT) coefficients. In a CWT, the wavelet corresponding to scale a and time location b is [3] :

$$\Psi_{a,b}(t) = a^{-1/2}\Psi\left(\frac{t-b}{a}\right) \quad , \qquad a \in \mathbf{R}^*,\ b \in \mathbf{R} \tag{1}$$

where $\Psi(t)$ is the mother wavelet. This wavelet can be thought of as a bandpass function which generates a continuous family of wavelets $\Psi_{a,b}$. Time t and time-scale parameters (a, b) vary continuously :

$$\mathrm{CWT}\{x(t)\ ;\ a\ ,\ b\} = \int x(t)\Psi^*_{a,\,b}(t)\mathrm{d}t \tag{2}$$

For WS coefficients, time is still continuous but time-scale parameters (b, a) are sampled on a dyadic grid in the time-scale plane (b, a). An usual definition of WS coefficients is :

$$C_{j,k} = \mathrm{CWT}\left\{x(t)\ ;\ a = 2^j,\ b = k2^j\right\} = \int x(t)\Psi_{j,k}(t)\mathrm{d}t \qquad (3)$$

where j, $k \in \mathbf{Z}$. The wavelets are in this case :

$$\Psi_{j,k}(t) = 2^{-j/2}\,\Psi(2^{-j}t - k) \qquad (4)$$

where j is the scaling parameter and k, the shifting one.

The WS scheme was termed the « wavelet series transform (WST) » in [6]. The direct transform DWST is defined by (3), while the inverse transform, IWST{$C_{j,k}$}, is defined by :

$$x(t) = \sum_{j\in\mathbf{Z}}\sum_{k\in\mathbf{Z}} C_{j,k}\Psi_{j,k}(t) \qquad (5)$$

2.2. Development of reconstruction method

Let us consider N_s nonuniform samples of a band-limited continuous-time random signal $x(t)$ sampled with a nonuniform sampling process $\pi = \{t_i \mid i = 0, \ldots, N_s\text{-}1\}$. The frequency bandwidth of the signal $x(t)$ is supposed known or an initial guess is made using autocorrelation information. The aim is to reconstruct $x(t)$ by interpolating its available samples $x_s = \{x(t_i)\}$ while preserving the frequency content of the signal.

The principle of the proposed reconstruction method involves application of a modified iterated DWST (3) to compute the $C_{j,k}$ coefficients, and then of an IWST (5) to reconstruct the continuous-time signal $x(t)$. The reconstructed continuous-time signal is called $y(t)$.

In this algorithm, a band-limited continuous-time Morlet wavelet is considered as mother wavelet. Morlet wavelet is obtained by modulating a Gaussian function. The Ψ function of mother wavelet is defined in this case by :

$$\Psi(t) = \exp(-\alpha\, t^2/2 + j\beta t) \qquad (6)$$

Figure 1 shows real and imaginary parts, and also magnitude of a Morlet wavelet.

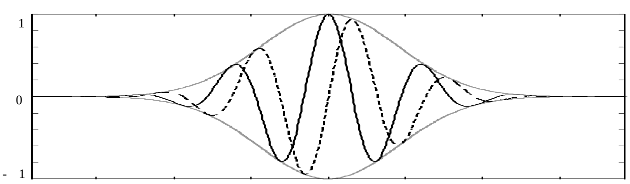


Figure 1. Real part (black line), imaginary part (doted line) and magnitude (gray line) of a Morlet wavelet

This algorithm neglects the analyzing wavelet contribution when its magnitude is less than a given minimum threshold, i.e. for time instants beyond a consequent time support of duration T_W. Therefore, the algorithm uses the modified iterated DWST to compute each $C_{j,k}$ coefficient, successively, in a least-squares sense.

Parameters α and β defining the Ψ function of mother wavelet, and also two supplementary parameters called τ and J must be chosen before starting the algorithm. Parameter τ represents the time shift interval between successive analyzing wavelets at the first resolution level (figure 2.a.). The maximum resolution level needed by the algorithm is called J (figure 2.b.).

The two parameters α and β are not independent but have to satisfy the numerical condition : $5 \le \beta\alpha^{-1/2} \le 6$ [1]. The choice of these parameters is made in order to assure a suitable spectral overlapping between successive resolution levels of the analyzing wavelets (figure 2.b.). Then, J is chosen to cover the whole frequency bandwidth of the original signal $x(t)$. Finally, τ is chosen to assure a good time overlapping between successive shifted analyzing wavelets belonging to the same resolution level. Figure 2 helps to understand the criteria for the choice of parameters α, β, J and τ.

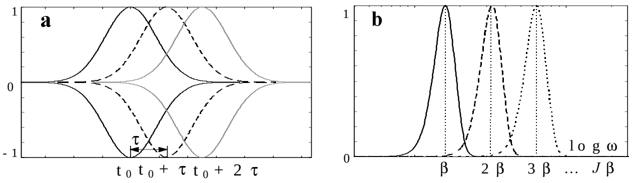


Figure 2. a. Time overlapping between successive wavelets at the same resolution level ; **b.** spectral overlapping between successive resolution levels ($\alpha = 0.025$, $\beta = 1$, $\tau = 2\pi$)

Let us call M_j the shifted wavelets number necessary to cover the whole signal x_s at each resolution level j. The algorithm computes the coefficients $C_{j,k}$ associated to the scaled (compressed) and shifted wavelets, and reconstructs the approximated continuous-time signal $y(t)$ and its derivatives :

1. **For** $j = 0 : \text{-}1 : \text{-}(J\text{-}1)$ (i.e. scaled wavelet level)
2. **For** $k = 0 : (M_j\text{-}1)$ (i.e. shifted wavelet)

   If $\exists$ at least N_{s_min} instants t_i, $t_i \in [k2^j\tau - 2^{j-1}T_W,\ k2^j\tau + 2^{j-1}T_W]$ **then**

$$x(t_i) = \mathrm{Re}(C_{j,k}\Psi_{j,k}(t_i)) \quad , \quad C_{j,k} \in \mathbf{C} \qquad (7)$$

$$\mathbf{x} = \mathrm{Re}(C_{j,k}\Psi_{j,k}) \ \Rightarrow\ \hat{C}_{j,k} \qquad (8)$$

$$\hat{\mathbf{x}} = \mathrm{Re}(\hat{C}_{j,k}\Psi_{j,k}) \qquad (9)$$

   If norm({ $\hat{x}(t_i)$ }) < norm({$x(t_i)$}) **then**

$$e(t_i) = x(t_i) - \hat{x}(t_i) = x(t_i) - \mathrm{Re}(\hat{C}_{j,k}\Psi_{j,k}(t_i)) \qquad (10)$$

$$x(t_i) = e(t_i) \qquad (11)$$

$$\hat{C}_{j,k\_new} = \hat{C}_{j,k\_old} + \hat{C}_{j,k} \qquad (12)$$

   else go to step 2

3. Iterate steps 1 and 2 until a given reconstruction error threshold ε is reached : norm({$x(t_i)$}) < ε.

4. Finally, reconstruct $y(t)$ with (5) and establish the formulas of all its derivatives $y^{(l)}(t)$, $l = 1, 2, \ldots$:

$$y(t) = \sum_{j\in\mathbf{Z}} \sum_{k\in\mathbf{Z}} \hat{C}_{j,k} \Psi_{j,k}(t) \qquad (13)$$

$$y^{(l)}(t) = \sum_{j\in\mathbf{Z}} \sum_{k\in\mathbf{Z}} \hat{C}_{j,k} \Psi^{(l)}_{j,k}(t) \quad , \qquad l = 1, 2, \ldots \qquad (14)$$

Figure 3 shows the pyramidal structure of the described algorithm.

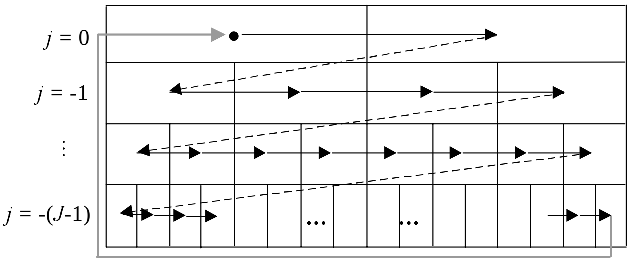


Figure 3. Pyramidal structure of iterated algorithm

More complicated scanning schemes have been considered to optimize computing time and memory needs : alternate scanning (left-right then right-left), level by level scanning (several iterations at each resolution level) etc. The convergence speed is different but, fortunately the final result is practically the same.

2.3. Discussion

The following discussion analyzes advantages of this reconstruction method and its computational requirements.

Since $\pi = \{t_i\}$ is a nonuniform sampling set, there are some sections where signal is densely sampled and others where it is sparsely sampled. In our algorithm, at each resolution level, only sections containing a given minimum number of samples N_{s_min} are considered. Consequently, this reconstruction method does not present the risk to introduce in the reconstructed signal $y(t)$ frequency components that are not actually present in the original signal $x(t)$.

The described method is characterized by a low computational cost. Each wavelet coefficient is computed independently from the others, as the best-matching solution of system (8) in a least-squares sense. This is a consequence of considering wavelet contribution negligible when its magnitude is less than a given minimum threshold.

In the algorithm, equation (10) defines the residual signal $\{e(t_i)\}$. Algorithm's convergence is guaranteed by taking into account (equation 12) only the coefficients minimizing the energy of residual signal at each shift step. As a consequence, the residual error is always decreasing between two consecutive iterations.

Choosing a wavelet basis as a windowing technique with variable-sized intervals to reconstruct a signal by interpolating its nonuniform samples is very advantageous. In fact, frequency resolution is high at low-frequency but at the price of a poor time resolution due to large dilation of analyzing wavelet. Inversely, compressing analyzing wavelet at high-frequency increases time resolution but decreases frequency one.

In the proposed method, a Morlet wavelet basis is considered. Generally, any band-limited continuous-time wavelet basis can be employed.

3. EXPERIMENTAL RESULTS

In our experiments we consider that continuous-time signals are nonuniformly sampled with a Poisson sampling process.

Firstly, we consider a continuous-time random signal $x(t)$ being the output of an analog linear filter having f_1 = 0.1 Hz and f_2 = 0.5 Hz as resonance frequencies and the transfer function :

$$H(p) = \frac{p^3 + 2p^2 + p}{(p^2 + 0.002p + 10)(p^2 + 0.002p + 0.4)} \qquad (15)$$

when its input is a Gaussian continuous-time white noise.

Figure 4 shows an example of our method capability to perform spectral analysis. It displays a comparison between the power spectral densities (PSD) of the original signal $x(t)$ and the reconstructed signal $y(t)$.

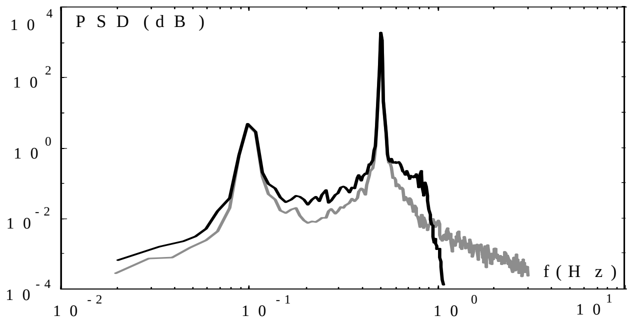


Figure 4. PSD of the original signal $x(t)$ (gray) and of the reconstructed signal $y(t)$ (black)

Secondly, we consider the case of a 6th order Butterworth lowpass analog filter with a cutoff frequency of 0.5 Hz, described by the transfer function :

$$H(p) = \frac{961.4}{p^6 + 12.1p^5 + 73.7p^4 + 283.4p^3 + 727.1p^2 + 1182.4p + 961.4} \qquad (16)$$

Figure 5 represents the residual relative error of the reconstruction method as a function of iterations number used in the algorithm.

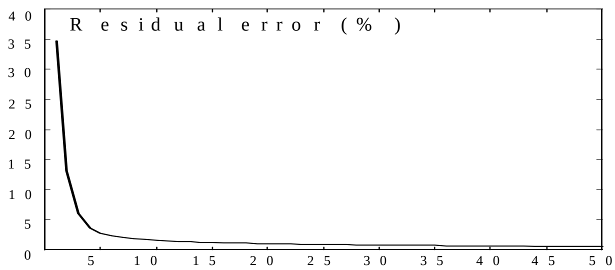
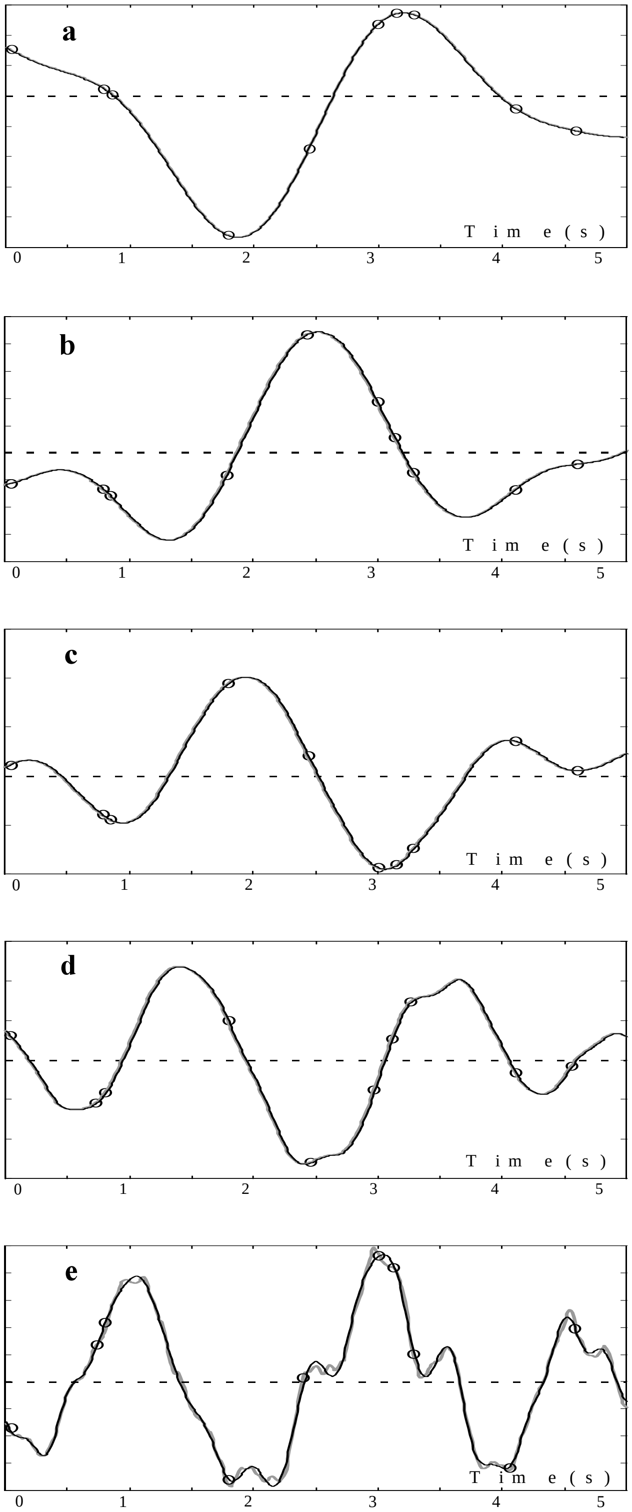


Figure 5. Residual relative error as a function of iterations number

Figures 6.a.-f. shows our method capability to reconstruct the original signal $x(t)$ and its derivatives until the 5th order. Sampling instants are marked with "o". It may be seen that the reconstruction error is low for the signal and its first three derivatives, and becomes important after the 4th one.



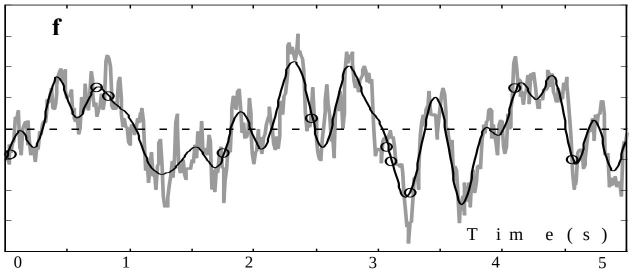


Figure 6. Signal and derivatives :
original ***x*** in gray and reconstructed ***y*** in black
a. $x(t)$ and y(t) **b.** $x^{(1)}(t)$ and $y^{(1)}(t)$ **c.** $x^{(2)}(t)$ and $y^{(2)}(t)$
d. $x^{(3)}(t)$ and $y^{(3)}(t)$ **e.** $x^{(4)}(t)$ and $y^{(4)}(t)$ **f.** $x^{(5)}(t)$ and $y^{(5)}(t)$

4. CONCLUSION AND FUTURE WORK

We proposed a wavelet analysis method to reconstruct continuous-time random signals when nonuniform samples are available. An advantage of the implemented algorithm is to perform reconstruction while preserving the frequency content of the original signal. The pyramidal structure of the algorithm allows to take advantage of nonuniform signal sampling by varying frequency and time resolutions in inverse ratio. Iterating the algorithm several times leads to a very small reconstruction error. The algorithm is characterized by low computational cost.

Future developments of this method concern its adaptation to the real-time reconstruction. Also, in our future work, the analytical formulas obtained for the reconstructed continuous-time signal and its derivatives will be used to perform a parametric estimation of continuous-time ARMA models classically adopted for continuous-time random signals modeling.

5. REFERENCES

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