NEW CONCEPTS IN NARROWBAND AND WIDEBAND WEYL CORRESPONDENCE TIME-FREQUENCY TECHNIQUES*

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ABSTRACT

We propose the *new* P_0-Weyl symbol to analyze system induced time shifts and scale changes on the input signal. This new Weyl symbol (WS) is useful in wideband signal analysis. We also propose *new* WS as tools for analyzing systems which produce dispersive frequency shifts on the signal. We obtain these generalized, frequency-shift covariant WS by warping conventional, narrowband WS. Using the new, generalized WS, we provide a formulation for the Weyl correspondence for linear systems with instantaneous frequency characteristics matched to user specified characteristics. We also propose a *new* interpretation of linear signal transformations as weighted superpositions of non-linear frequency shifts on the signal. Application examples in signal analysis and detection demonstrate the advantages of our new results.

1. INTRODUCTION

Time-frequency (TF) formulations of the conventional Weyl symbol (WS) and its 2-D Fourier transform (FT), the spreading function (SF), have been successfully used in the analysis of linear time-varying systems and nonstationary processes [5, 12, 6]. The conventional WS and SF are defined[1], respectively, as [5]

$$\mathrm{WS}_{\mathcal{L}}(t,f) = \int K_{\mathcal{L}}(t+\frac{\tau}{2}, t-\frac{\tau}{2})\, e^{-j2\pi f\tau} d\tau, \qquad (1)$$

$$\mathrm{SF}_{\mathcal{L}}(\tau,\nu) = \int K_{\mathcal{L}}(t+\frac{\tau}{2}, t-\frac{\tau}{2})\, e^{-j2\pi t\nu} dt, \qquad (2)$$

for an operator $\mathcal{L}$ on $\mathrm{L}_2(\mathbb{R})$ with operator kernel $K_{\mathcal{L}}(t,\tau)$ [2]. The WS can be interpreted as the transfer function of a time-varying system or as the time-varying spectrum of a random process. The Weyl correspondence is the 2-D inner product of the Wigner distribution (WD) [3] of a random process $x(t)$ and the WS of $\mathcal{L}$,

$$\int (\mathcal{L}x)(t)\, x^*(t) dt = \int\int \mathrm{WD}_x(t,f)\, \mathrm{WS}_{\mathcal{L}}(t,f)\, dt\, df. \qquad (3)$$

This important relationship provides a definition of a TF concentration measure [12], and is useful in TF detection [7, 11] and analysis [8] applications. The SF provides an important interpretation of a time-varying system output as a weighted superposition of time shifts $(\mathcal{S}_\tau x)(t) = x(t-\tau)$, and frequency shifts $(\mathcal{M}_\nu x)(t) = e^{j2\pi\nu t}x(t)$ on the input signal $x(t)$, where the weight is the SF [12], i.e. $(\mathcal{L}x)(t) = \int\int \mathrm{SF}_{\mathcal{L}}(\tau,\nu) e^{-j\pi\tau\nu} (\mathcal{M}_\nu \mathcal{S}_\tau x)(t) d\tau d\nu$. This is comparable to the conventional interpretation of the (convolution) output of a linear time-invariant filter as a weighted superposition of time shifts on the input signal. The support region of the SF has been used to define underspread random processes [6], a useful concept in detection applications [7].

In [12], the narrowband Weyl correspondence in (3) was modified for *affine* processes using a wideband SF (WSF). It characterizes a wideband system output as a weighted superposition of time shifts and scale changes on the input signal, where the weight is the WSF. However, no wideband WS was given; it is needed to yield information on the actual TF structure of the output of a wideband system. Thus, we propose a *new* WS, the P_0-Weyl symbol (P_0WS), for a system that produces time shifts and scale changes. The P_0WS is important as it is a time-shift and scale covariant TF representation that is a natural extension of the conventional WS to wideband processes. The inner product of this *new* P_0WS with the unitary Bertrand P_0-distribution provides an alternative TF formulation of the affine Weyl correspondence in [12].

The conventional WS and SF are no longer adequate to characterize linear systems whose nonstationary process is not matched to simple time and frequency shifts. Thus, in this paper, we propose *new* WS as tools for analyzing systems which produce dispersive frequency shifts on the signal. These new TF WS are important since they can be interpreted as time-varying transfer functions for such systems. We derive such generalized WS by warping the conventional narrowband WS. We provide a TF formulation for the Weyl correspondence for linear systems with instantaneous frequency characteristics matched to a specified warping. Examples will be given to demonstrate how the generalized WS greatly simplifies when matched to the system. We also extend these results by generalizing the P_0WS to analyze systems that produce non-linear time shifts. Analysis and detection application examples demonstrate the importance of these new TF techniques.

2. WIDEBAND FORMULATION OF THE WEYL SYMBOL

The affine version of the Weyl correspondence proposed in [12],

$$\int (\mathcal{B}X)(f)\, X^*(f) df = \int\int \mathrm{WSF}_{\mathcal{B}}(\tau,\alpha) \mathrm{WAF}^*_X(\tau,\alpha) d\tau d\alpha, \qquad (4)$$

is written in terms of the wideband ambiguity function (WAF) [12] and the wideband SF,

$$\mathrm{WSF}_{\mathcal{B}}(\tau,\alpha) = \int_0^\infty f\, \Gamma_{\mathcal{B}}(f\lambda(\alpha)e^{\frac{\alpha}{2}}, f\lambda(\alpha)e^{-\frac{\alpha}{2}})\lambda(\alpha) e^{j2\pi f\tau} df, \qquad (5)$$

of the operator $\mathcal{B}$ whose frequency domain kernel function is $\Gamma_{\mathcal{B}}(f,\nu)$. Here, $\lambda(\alpha) = \frac{\alpha/2}{\sinh \alpha/2}$, and $X(f) = \mathcal{F}_{t\to f}\{x(t)\}$ is the FT of $x(t)$. The output of a linear system can now be interpreted as a weighted superposition of time shifts and scale changes on the input signal. The weights are the WSF in (5), i.e.

$$(\mathcal{B}X)(f) = \int\int \mathrm{WSF}_{\mathcal{B}}(\tau,\alpha)\ \mathcal{F}_{t\to f}\{(\mathcal{C}_{e^\alpha} \mathcal{S}_{\tilde{\tau}_\alpha} x)(t)\}\, d\tau\, d\alpha$$

with time-shift $\tilde{\tau}_\alpha = \tau e^{\alpha/2}/\lambda(\alpha)$. However, the WSF provides information only on relative time lags and scale changes of a process. It is desirable to have an affine WS that gives the actual TF structure of processes which cause time shifts and scale changes.

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[1] Unless otherwise specified, integrals range from $-\infty$ to ∞.

$\xi(b)$	Weyl Symbol (WS) time-frequency representation	Spreading Function (SF)
1 to 1	$\mathrm{GWS}_{\mathcal{Y}}(t,f) = \mathrm{WS}_{\mathcal{W}_\xi \mathcal{Y} \mathcal{W}_\xi^{-1}}(t_r \xi(\frac{t}{t_r}), \frac{f}{t_r \varphi(t)})$ (generalized WS in (9))	$\mathrm{GSF}_{\mathcal{Y}}(\zeta,\beta) = \mathrm{SF}_{\mathcal{W}_\xi \mathcal{Y} \mathcal{W}_\xi^{-1}}(t_r\zeta, \frac{\beta}{t_r})$ in (10)
b	$\mathrm{WS}_{\mathcal{L}}(t,f) = \int K_{\mathcal{L}}(t+\frac{\tau}{2}, t-\frac{\tau}{2}) e^{-j2\pi f\tau} d\tau$ (conventional WS in (1))	$\mathrm{SF}_{\mathcal{L}}(\tau,\nu) = \int K_{\mathcal{L}}(t+\frac{\tau}{2}, t-\frac{\tau}{2}) e^{-j2\pi t\nu} dt$ in (2)
$\xi_{\ln}(b)$	$\mathrm{HWS}_{\mathcal{Y}}(t,f) = \mathrm{WS}_{\mathcal{W}_{\xi_{\ln}} \mathcal{Y} \mathcal{W}_{\xi_{\ln}}^{-1}}(t_r \ln(\frac{t}{t_r}), \frac{tf}{t_r})$ (hyperbolic WS in (7))	$\mathrm{HSF}_{\mathcal{Y}}(\zeta,\beta) = \mathrm{SF}_{\mathcal{W}_{\xi_{\ln}} \mathcal{Y} \mathcal{W}_{\xi_{\ln}}^{-1}}(t_r\zeta, \frac{\beta}{t_r})$ in (8)
$\xi_\kappa(b)$	$\mathrm{PWS}_{\mathcal{Y}}^{(\kappa)}(t,f) = \mathrm{WS}_{\mathcal{W}_{\xi_\kappa} \mathcal{Y} \mathcal{W}_{\xi_\kappa}^{-1}}(t_r \xi_\kappa(\frac{t}{t_r}), \frac{f}{t_r \varphi_\kappa(t)})$ (power WS)	$\mathrm{PSF}_{\mathcal{Y}}^{(\kappa)}(\zeta,\beta) = \mathrm{SF}_{\mathcal{W}_{\xi_\kappa} \mathcal{Y} \mathcal{W}_{\xi_\kappa}^{-1}}(t_r\zeta, \frac{\beta}{t_r})$
$\xi_{\exp}(b)$	$\mathrm{EWS}_{\mathcal{Y}}(t,f) = \mathrm{WS}_{\mathcal{W}_{\xi_{\exp}} \mathcal{Y} \mathcal{W}_{\xi_{\exp}}^{-1}}(t_r e^{t/t_r}, f e^{-t/t_r})$ (exponential WS)	$\mathrm{ESF}_{\mathcal{Y}}(\zeta,\beta) = \mathrm{SF}_{\mathcal{W}_{\xi_{\exp}} \mathcal{Y} \mathcal{W}_{\xi_{\exp}}^{-1}}(t_r\zeta, \frac{\beta}{t_r})$

Table 1: *Various Weyl symbols and spreading functions for a given warping function $\xi(b)$. Here, $\mathcal{Y}$ is defined based on the domain of $\xi(b)$. For example, for the HWS, $\mathcal{Y}$ is defined on $L_2(\mathbb{R}^+)$. The warping operator is $(\mathcal{W}_\xi x)(t) = x(t_r \xi^{-1}(\frac{t}{t_r}))/|t_r \varphi(t_r \xi^{-1}(\frac{t}{t_r}))|^{1/2}$ and $(\mathcal{W}_\xi \mathcal{W}_\xi^{-1} x)(t) = x(t)$. Here, $\xi_{\ln}(b) = \ln(b)$, $\xi_\kappa(b) = \mathrm{sgn}(b)|b|^\kappa$, $\xi_{\exp}(b) = e^b$, $\varphi(t) = \frac{d}{dt}\xi(\frac{t}{t_r})$, and $\varphi_\kappa(t) = \frac{d}{dt}\xi_\kappa(\frac{t}{t_r})$.*

In this paper, we propose a *new* affine WS, the P_0-Weyl symbol (P_0WS), that we define for $f > 0$ as

$$\mathrm{P}_0\mathrm{WS}_{\mathcal{B}}(t,f) = f \int \Gamma_{\mathcal{B}}(f\lambda(\alpha)e^{\frac{\alpha}{2}}, f\lambda(\alpha)e^{-\frac{\alpha}{2}})\lambda(\alpha)e^{j2\pi f t\alpha} d\alpha, \quad (6)$$

for a given operator $\mathcal{B}$ on $L_2(\mathbb{R}^+)$. The affine Weyl correspondence in (4) can now be alternatively expressed,

$$\int (\mathcal{B}X)(f)\, X^*(f) df = \int\int \mathrm{P}_0\mathrm{WS}_{\mathcal{B}}(t,f)\, \mathrm{P}_{0_X}(t,f)\, dt\, df,$$

in terms of the P_0WS and the unitary Bertrand P_0-distribution [1], $\mathrm{P}_{0_X}(t,f) = f\int X(f\lambda(\alpha)e^{\frac{\alpha}{2}}) X^*(f\lambda(\alpha)e^{-\frac{\alpha}{2}})\lambda(\alpha)e^{j2\pi t f\alpha} d\alpha$. The P_0WS is an intuitive tool for analyzing linear systems which produce time shifts and scale changes on the signal, and hence ideal for wideband TF system analysis. The modified 2-D FT relation between the P_0WS and the WSF in (5) is

$$\mathrm{P}_0\mathrm{WS}_{\mathcal{B}}(t,f) = \mathcal{F}_{\tau\to f}\{\mathcal{F}^{-1}_{\alpha\to tf}\{\mathrm{WSF}_{\mathcal{B}}(\tau,\alpha)\}\},$$

with inverse FT $\mathcal{F}^{-1}$. Some special cases of the P_0WS follow. If the operator output is the product $(\mathcal{B}X)(f) = X(f)H(f)$, whose inverse FT corresponds to convolution, i.e. a weighted superposition of time shifts operating on $x(t)$, then $\mathrm{P}_0\mathrm{WS}_{\mathcal{B}}(t,f) = H(f)$ is a TF transfer function dependent only on frequency. On the other hand, if the operator output is a weighted superposition of scale changes on $X(f)$, i.e. $(\mathcal{B}X)(f) = \int_0^\infty H(\nu) X(\frac{f}{t_r \nu}) \frac{d\nu}{\nu\sqrt{t_r}}$, $f > 0$, then the P_0WS is equal to the Mellin transform [1] of $H(f)$. Here, $t_r > 0$ is a fixed reference time. Thus, the P_0WS provides a concise, intuitive formulation of time-invariant or wideband systems.

3. GENERALIZATION OF NARROWBAND WEYL CORRESPONDENCE

3.1. Hyperbolic Weyl Symbol and Spreading Function

If a system imposes hyperbolic frequency shifts and scale changes on the input signal, *new* WS and *new* SF are needed for analysis. The TF geometry of these new WS and SF should reflect the hyperbolic system changes on the input signal. Thus, for an operator $\mathcal{Y}$ on $L_2(\mathbb{R}^+)$ with kernel $K_{\mathcal{Y}}(t,\tau)$, we define the hyperbolic WS and SF, respectively, as

$$\mathrm{HWS}_{\mathcal{Y}}(t,f) = t\int_0^\infty K_{\mathcal{Y}}(te^{\zeta/2}, te^{-\zeta/2})\, e^{-j2\pi t f\zeta} d\zeta, \; t > 0 \quad (7)$$

$$\mathrm{HSF}_{\mathcal{Y}}(\zeta,\beta) = \int_0^\infty K_{\mathcal{Y}}(te^{\zeta/2}, te^{-\zeta/2})\, e^{-j2\pi\beta\ln(\frac{t}{t_r})} dt. \quad (8)$$

The relation between the HWS and the HSF is given as

$$\mathrm{HSF}_{\mathcal{Y}}(\zeta,\beta) = \mathcal{F}^{-1}_{\gamma\to\zeta}\{\mathcal{P}_{t\to\beta}\{\mathrm{HWS}_{\mathcal{Y}}(t,\gamma/t)\}\}$$

where $\mathcal{P}_{t\to\beta}\{x(t)\} = \int_0^\infty x(t) e^{-j2\pi\beta\ln\frac{t}{t_r}} \frac{dt}{t} = \rho_x^{(\xi_{\ln})}(\beta)$, $t > 0$, is a version of the Mellin transform [1]. Note the similarities between the conventional WS (and SF) and the HWS (and HSF) summarized in Table 1. Row 4 shows that the hyperbolic WS in (7) can be obtained from the conventional WS in (1) by first unitarily warping the operator $\mathcal{Y}$ and then transforming the TF axes. For the HSF, the axes are simply scaled since they show only relative TF lags, not absolute TF locations.

The Weyl correspondence in (3) can now be written in terms of the HWS and $\mathrm{Q}_x(t,f)$, the Altes-Marinovic Q-distribution [9],

$$\int_0^\infty (\mathcal{Y}x)(t)\, x^*(t) dt = \int_0^\infty \int_{-\infty}^\infty \mathrm{HWS}_{\mathcal{Y}}(t,f)\, \mathrm{Q}_x(t,f)\, dt\, df.$$

This *new* form of the Weyl correspondence may be useful in detection applications of nonstationary processes and systems with hyperbolic instantaneous frequency characteristics. These formulations are important as they provide a new interpretation of these non-linear system outputs as weighted superpositions of hyperbolic frequency shifts and scale changes on the input signal, i.e.

$$(\mathcal{Y}x)(t) = \int\int \mathrm{HSF}_{\mathcal{Y}}(\zeta,\beta)\, e^{-j\pi\zeta\beta}\, (\mathcal{H}_\beta \mathcal{C}_{e^\zeta} x)(t)\, d\zeta\, d\beta$$

where $(\mathcal{H}_\beta x)(t) = e^{j2\pi\beta\ln(\frac{t}{t_r})} x(t)$ is the hyperbolic shift operator and $(\mathcal{C}_a x)(t) = x(\frac{t}{a})/|a|^{\frac{1}{2}}$ is the scaling operator. Thus, $\mathrm{HSF}_{\mathcal{Y}}$ weighs the relative importance of hyperbolic frequency shifts and scale changes caused by a linear system. In rows 5-7 of Table 2, we provide examples of systems/operators well-matched to concise HSF and HWS TF representations. For example, if the system output is the scale convolution of the input signal $x(t)$ and a function $g(t)$ (column 4, row 6), then the HWS in (7) of the operator is the Mellin transform of $g(t)$, i.e. $\mathrm{HWS}_{\mathcal{Y}}(t,f) = \rho_g^{(\xi_{\ln})}(tf)$ (column 2, row 6), which is intuitive as the Mellin is a natural transform for scale operations. For comparison, the conventional WS in (1), $\mathrm{WS}_{\mathcal{Y}}(t,f) = \int_0^\infty \frac{\sqrt{t_r}}{t-\tau/2} g(t_r \frac{t+\tau/2}{t-\tau/2}) e^{-j2\pi\tau f} d\tau$, of the *same* operator is difficult to interpret. In Section 5, we provide applications to demonstrate the importance of the HWS.

3.2. Power Weyl Symbol and Spreading Function

We obtain the κth power WS ($\mathrm{PWS}^{(\kappa)}$) and the κth power SF ($\mathrm{PSF}^{(\kappa)}$), for an operator $\mathcal{Y}$ on $L_2(\mathbb{R})$, by warping the conventional WS and SF as shown in row 5 of Table 1. The relation between $\mathrm{PWS}^{(\kappa)}$ and $\mathrm{PSF}^{(\kappa)}$ is given by

$$\mathrm{PSF}_{\mathcal{Y}}^{(\kappa)}(\zeta,\beta) = \mathcal{F}^{-1}_{\gamma\to\zeta}\{\mathcal{P}^{(\kappa)}_{t\to\beta}\{\mathrm{PWS}_{\mathcal{Y}}^{(\kappa)}(t,\gamma\varphi_\kappa(t))\}\},$$

where $\mathcal{P}^{(\kappa)}_{t\to\beta}\{x(t)\} = \int x(t) e^{-j2\pi\beta\,\mathrm{sgn}(t)|\frac{t}{t_r}|^\kappa} \frac{\kappa}{t_r}|\frac{t}{t_r}|^{\kappa-1} dt$. The Weyl correspondence can now be expressed [4] in terms of $\mathrm{PWS}^{(\kappa)}$ and a power warped version of the WD [10]. The operator output can be interpreted as a weighted superposition of κth power frequency shifts on the input signal with weights $\mathrm{PSF}_{\mathcal{Y}}^{(\kappa)}$, i.e.

$$(\mathcal{Y}x)(t) = \int\int \mathrm{PSF}_{\mathcal{Y}}^{(\kappa)}(\zeta,\beta)\, e^{-j\pi\zeta\beta} e^{j2\pi\beta\,\mathrm{sgn}(t)|\frac{t}{t_r}|^\kappa}\, \tilde{x}_{\zeta,\kappa}(t)\, d\zeta\, d\beta$$

where $\tilde{x}_{\zeta,\kappa}(t) = |1 - \mathrm{sgn}(t)\zeta|\frac{t_r}{t}|^\kappa|^{\frac{1-\kappa}{2\kappa}} x(t|1 - \mathrm{sgn}(t)\zeta|\frac{t_r}{t}|^\kappa|^{\frac{1}{\kappa}})$. An important fact is that when $\kappa = 1$, the $\mathrm{PWS}^{(\kappa)}$ and $\mathrm{PSF}^{(\kappa)}$ simplify to the conventional WS and SF in (1) and (2), respectively.

Cases	WS, $\mathrm{GWS}_{\mathcal{Y}}(t,f)$	Kernel, $K_{\mathcal{Y}}(t,\tau)$	Operator output, $(\mathcal{Y}x)(t)$	SF, $\mathrm{GSF}_{\mathcal{Y}}(\zeta,\beta)$
Conventional [12]	$h(t)$	$h(\tau)\delta(t-\tau)$	$h(t)x(t)$	$\mathrm{H}(\beta/t_r)$
	$G(f)$	$g(t-\tau)$	$\int g(t-\tau)x(\tau)d\tau$	$g(t_r\zeta)\delta(\beta/t_r)$
	$h(t)G(f)$	$h(\frac{t+\tau}{2})g(t-\tau)$	$\int h(\frac{t+\tau}{2})g(t-\tau)x(\tau)d\tau$	$g(t_r\zeta)\mathrm{H}(\beta/t_r)$
Hyperbolic	$\sqrt{t}h(t)$	$\sqrt{\tau}h(\tau)\delta(t-\tau)$	$\sqrt{t}h(t)x(t)$	$\rho_h^{(\xi_{\ln})}(\beta)\delta(\zeta)$
	$\rho_g^{(\xi_{\ln})}(tf)$	$\frac{\sqrt{t_r}}{\tau}g(t_r\frac{t}{\tau})$	$\int_0^\infty \frac{\sqrt{t_r}}{\tau}g(t_r\frac{t}{\tau})x(\tau)d\tau$	$\sqrt{t_r e^\zeta}g(t_r e^\zeta)\delta(\beta)$
	$\sqrt{t}h(t)\rho_g^{(\xi_{\ln})}(tf)$	$\frac{h(\sqrt{t\tau})}{\sqrt[4]{t\tau}}\sqrt{\frac{t_r t}{\tau}}g(t_r\frac{t}{\tau})$	$\int_0^\infty \frac{h(\sqrt{t\tau})}{\sqrt[4]{t\tau}}\sqrt{\frac{t_r t}{\tau}}g(t_r\frac{t}{\tau})x(\tau)d\tau$	$\sqrt{t_r e^\zeta}g(t_r e^\zeta)\rho_h^{(\xi_{\ln})}(\beta)$
Generalized	$\frac{h(t)}{\lvert\varphi(t)\rvert^{1/2}}$	$\frac{h(\tau)}{\lvert\varphi(\tau)\rvert^{1/2}}\delta(t-\tau)$	$\frac{h(t)}{\lvert\varphi(t)\rvert^{1/2}}x(t)$	$\rho_h^{(\xi)}(\beta)\delta(\zeta)$
	$\rho_g^{(\xi)}(\frac{f}{\varphi(t)})$	$\lvert\frac{\varphi(t)\varphi(\tau)}{\varphi(\hat{t})}\rvert^{1/2}g(\hat{t})$	$\int_p^q \lvert\frac{\varphi(t)\varphi(\tau)}{\varphi(\hat{t})}\rvert^{1/2}g(\hat{t})x(\tau)d\tau$	$g(t_r\xi^{-1}(\zeta))\frac{\delta(\beta)}{\lvert\varphi(t_r\xi^{-1}(\zeta))\rvert^{1/2}}$
	$\frac{h(t)}{\lvert\varphi(t)\rvert^{1/2}}\rho_g^{(\xi)}(\frac{f}{\varphi(t)})$	$\lvert\frac{\varphi(t)\varphi(\tau)}{\varphi(\hat{t})}\rvert^{1/2}g(\hat{t})\frac{h(\tilde{t})}{\lvert\varphi(\tilde{t})\rvert^{1/2}}$	$\int_p^q \lvert\frac{\varphi(t)\varphi(\tau)}{\varphi(\hat{t})}\rvert^{1/2}g(\hat{t})\frac{h(\tilde{t})}{\lvert\varphi(\tilde{t})\rvert^{1/2}}x(\tau)d\tau$	$\frac{g(t_r\xi^{-1}(\zeta))}{\lvert\varphi(t_r\xi^{-1}(\zeta))\rvert^{1/2}}\rho_h^{(\xi)}(\beta)$

Table 2: *Examples of various Weyl symbols and their corresponding kernels, operator outputs, and spreading functions. Here,* $\rho_h^{(\xi)}(\beta) = \int_p^q h(t)\sqrt{\varphi(t)}e^{-j2\pi\beta\xi(\frac{t}{t_r})}dt$ *is a normalized generalized transform where* $\varphi(t) = \frac{d}{dt}\xi(\frac{t}{t_r})$ *and the integration range* $[p,q]$ *is determined by the domain of* $\xi(b)$. *The generalized transform simplifies to the Fourier transform when* $\xi(b)=b$ *and to the Mellin transform,* $\rho^{(\xi_{\ln})}(\beta)$, *when* $\xi(b)=\ln b$. *Here,* $\hat{t} = t_r\xi^{-1}(\xi(\frac{t}{t_r}) - \xi(\frac{\tau}{t_r}))$, $\tilde{t} = t_r\xi^{-1}(\frac{\xi(\frac{t}{t_r})+\xi(\frac{\tau}{t_r})}{2})$, $H(f)=\mathcal{F}_{t\to f}\{h(t)\}$, *and* $G(f)=\mathcal{F}_{t\to f}\{g(t)\}$.

3.3. Exponential Weyl Symbol and Spreading Function

Using the warping $\xi_{\exp}(b) = e^b$ in row 6 of Table 1, we obtain the exponential WS (EWS) and the exponential SF (ESF) for $\mathcal{Y}$ on $L_2(\mathbb{R})$. Two transforms link $\mathrm{EWS}_{\mathcal{Y}}(t,f)$ and $\mathrm{ESF}_{\mathcal{Y}}(\zeta,\beta)$,

$$\mathrm{ESF}_{\mathcal{Y}}(\zeta,\beta) = \mathcal{F}^{-1}_{\gamma\to\zeta}\{\mathcal{E}_{t\to\beta}\{\mathrm{EWS}_{\mathcal{Y}}(t,\gamma e^{t/t_r}/t_r)\}\}.$$

Here, $\mathcal{E}_{t\to\beta}\{x(t)\} = \int x(t)e^{-j2\pi\beta e^{t/t_r}}e^{t/t_r}dt/t_r$. The exponential Weyl correspondence uses the EWS and an exponential warped WD [4, 10]. Operator output can be interpreted as a weighted superposition of exponential frequency shifts on the input signal, i.e.

$$(\mathcal{Y}x)(t) = \int\int \mathrm{ESF}_{\mathcal{Y}}(\zeta,\beta)\, e^{-j\pi\zeta\beta}\, e^{j2\pi\beta e^{t/t_r}}\, \tilde{x}_\zeta(t)\, d\zeta\, d\beta$$

where $\tilde{x}_\zeta(t) = [e^{t/t_r}/(e^{t/t_r} - \zeta)]^{1/2} x(t_r \ln(e^{t/t_r} - \zeta))$.

3.4. Generalized Weyl Symbol and Spreading Function

If a system imposes TF operators different from simple time or frequency shifts on the input signal, then new WS and SF are needed for analysis to reflect the dispersive changes on the input signal. We obtain the *new* generalized WS (GWS) and the *new* generalized SF (GSF) of an operator $\mathcal{Y}$ representing a system whose input signal is shifted in frequency in a non-linear manner related to a one-to-one warping function $\xi(b)$. The *new* GWS is defined as

$$\mathrm{GWS}_{\mathcal{Y}}(t,f) = \int K_{\mathcal{Y}}\left(t_r\Xi(\xi(\frac{t}{t_r}),\zeta),\ t_r\Xi(\xi(\frac{t}{t_r}),-\zeta)\right) \cdot \lvert\varphi(\Xi(\xi(\frac{t}{t_r}),\zeta))\varphi(\Xi(\xi(\frac{t}{t_r}),-\zeta))\rvert^{-1/2} e^{-j2\pi f\zeta/\varphi(t)} d\zeta, \quad (9)$$

where $\Xi(c,\zeta) = \xi^{-1}(c+\frac{\zeta}{2})$, $\xi^{-1}(\xi(b)) = b$, and $\varphi(t) = \frac{d}{dt}\xi(\frac{t}{t_r})$. Here, $K_{\mathcal{Y}}(t,\tau)$ is the kernel of the operator[2] $\mathcal{Y}$. The GWS preserves generalized frequency shifts on a random process $x(t)$, i.e.

$$y(t) = x(t)e^{j2\pi c\xi(\frac{t}{t_r})} \Rightarrow \mathrm{GWS}_{\mathbf{R}_y}(t,f) = \mathrm{GWS}_{\mathbf{R}_x}(t, f - c\varphi(t)),$$

where $\mathbf{R}_y$ and $\mathbf{R}_x$ are the correlation operators of $y(t)$ and $x(t)$, respectively. The *new* GSF is

$$\mathrm{GSF}_{\mathcal{Y}}(\zeta,\beta) = \int K_{\mathcal{Y}}(t_r\Xi(c,\zeta), t_r\Xi(c,-\zeta)) \cdot \lvert\varphi(\Xi(c,\zeta))\varphi(\Xi(c,-\zeta))\rvert^{-1/2} e^{-j2\pi c\beta} dc. \quad (10)$$

The integration limits in (9) and (10) are determined by the range of $\xi(b)$. The relation between $\mathrm{GWS}_{\mathcal{Y}}$ and $\mathrm{GSF}_{\mathcal{Y}}$ is

$$\mathrm{GSF}_{\mathcal{Y}}(\zeta,\beta) = \mathcal{F}^{-1}_{\gamma\to\zeta}\{\mathcal{G}_{t\to\beta}\{\mathrm{GWS}_{\mathcal{Y}}(t,\gamma\varphi(t))\}\}$$

where $\mathcal{G}_{t\to c}\{x(t)\} = \int x(t)e^{-j2\pi c\xi(\frac{t}{t_r})}\lvert\varphi(t)\rvert dt$ is a generalized transform dependent on the warping function $\xi(b)$.

The Weyl correspondence in (3) can now be expressed in terms of the *generalized* WS, $\mathrm{GWS}_{\mathcal{Y}}$,

$$\int (\mathcal{Y}x)(t)\, x^*(t) dt = \int\int \mathrm{GWD}_x(t,f)\, \mathrm{GWS}_{\mathcal{Y}}(t,f)\, dt\, df.$$

Here, $\mathrm{GWD}_x(t,f)$ is the generalized warped version of the WD [10] that depends on $\xi(b)$. This generalized form of the Weyl correspondence may be useful in detection applications of systems with arbitrary instantaneous frequency characteristics.

The operator output can now be interpreted as a weighted superposition of dispersive frequency shifts on the input

$$(\mathcal{Y}x)(t) = \int\int \mathrm{GSF}_{\mathcal{Y}}(\zeta,\beta)\, e^{-j\pi\zeta\beta}\, (\mathcal{D}_\beta \tilde{S}_\tau x)(t)\, d\zeta\, d\beta \quad (11)$$

where $(\mathcal{D}_\beta x)(t) = e^{j2\pi\beta\xi(\frac{t}{t_r})}x(t)$ is the generalized frequency-shifted signal, and $\tilde{S}_\tau = \mathcal{W}_\xi^{-1}\mathcal{S}_\tau\mathcal{W}_\xi$ is a generalized warped time-shift operator which can be further simplified depending on the specific warping function, $\xi(b)$.

In rows 8-10 of Table 2, we provide some examples of simple and intuitive GWS. In row 8, column 4, the operator in (11), $(\mathcal{Y}x)(t) = x(t)h(t)/\sqrt{\lvert\varphi(t)\rvert}$, windows the time domain signal $x(t)$. The generalized transform of this operator output intuitively results in the weighted superposition of dispersive frequency shifts $\mathcal{G}_{t\to c}\{(\mathcal{Y}x)(t)\} = \int h(t)x(t)e^{-j2\pi c\xi(\frac{t}{t_r})}\sqrt{\lvert\varphi(t)\rvert}dt$, and the GWS is simply the window, $\mathrm{GWS}_{\mathcal{Y}}(t,f) = h(t)/\sqrt{\lvert\varphi(t)\rvert}$. Depending on the choice of $\xi(b)$, all the WS and SF in Section 3 and Tables 1 and 2 are special cases of the GWS in (9) and GSF in (10). For example, the GWS examples in rows 8-10 in Table 2 simplify to the conventional WS examples in rows 2-4 when $\xi(b) = b$.

4. GENERALIZATION OF THE WIDEBAND P_0-WEYL SYMBOL

In Section 3, we generalized the conventional WS in (1) for systems that were better matched to unitarily warped time-shift and frequency-shift operators. For wideband systems, where it may be more intuitive to deal with unitarily warped time-shift and scale operators, we now generalize the wideband P_0WS in Section 2. For an operator $\mathcal{Q}$ representing a system whose input is time-shifted in a non-linear manner dependent on a one-to-one function $\theta(b)$, we define the generalized P_0WS as

[2] $\mathcal{Y}$ is defined on $L_2([\alpha,\beta])$; $[\alpha,\beta]$ depend on the domain of $\xi(b)$.

$$\mathrm{GP_0WS}_{\mathcal{Q}}(t,f)=|\theta(t_rf)|\int|\mu(\Theta(f,\alpha))\mu(\Theta(f,-\alpha))|^{-\frac{1}{2}}$$
$$\cdot\Gamma_{\mathcal{Q}}(\Theta(f,\alpha),\Theta(f,-\alpha))\lambda(\alpha)e^{j2\pi\frac{t\alpha}{\mu(f)}\theta(t_rf)}d\alpha$$
$$=\mathrm{P_0WS}_{\mathcal{U}_\theta\mathcal{Q}\mathcal{U}_\theta^{-1}}(\frac{t_rt}{\mu(f)},\theta(t_rf)/t_r) \qquad (12)$$

where $(\mathcal{U}_\theta X)(f) = X(\theta^{-1}(t_rf)/t_r)/\sqrt{|\mu(\theta^{-1}(t_rf)/t_r)|/t_r}$, $\Theta(f,\alpha) = \theta^{-1}(\theta(t_rf)\lambda(\alpha)e^{\alpha/2})/t_r$, and $\mu(f) = \frac{d}{df}\theta(t_rf)$. Here, the operator $\mathcal{Q}$ is defined on $L_2([p,q])$ where $[p,q]$ is determined by the domain of $\theta(b)$. Note that (12) shows the relationship to the $\mathrm{P_0WS}$ in (6). The corresponding generalized version of the affine SF is $\mathrm{GWSF}_{\mathcal{Q}}(\zeta,\alpha)=t_r\mathrm{WSF}_{\mathcal{U}_\theta\mathcal{Q}\mathcal{U}_\theta^{-1}}(t_r\zeta,\alpha)$.

5. APPLICATION EXAMPLES

• **Analysis problem:** In order to demonstrate the importance of the new generalized WS, we analyze a hyperbolic random process $x(t)=\sum_{i=1}^{3}\alpha_i x_i(t)$. Here, α_i are uncorrelated, zero-mean random weights and $x_i(t) = e^{j2\pi c_i \ln(t/t_r)}$, $t>0$, $i=1,2,3$, are hyperbolic FM, deterministic signals. Note that each signal term $x_i(t)$ has hyperbolic instantaneous frequency, c_i/t. One can show that the hyperbolic WS in (7) of the correlation operator $\mathbf{R}_x$ with kernel $K_{\mathbf{R}_x}(t,\tau) = \mathbf{E}[\,x(t)x^*(\tau)\,]$ simplifies to

$$\begin{aligned}\mathrm{HWS}_{\mathbf{R}_x}(t,f) &= \textstyle\sum_1^3 \mathbf{E}[\,|\alpha_i|^2]\ \mathrm{HWS}_{\mathbf{R}_{x_i}}(t,f)\\ &= \textstyle\sum_1^3 \mathbf{E}[\,|\alpha_i|^2]\ \delta(f-c_i/t)\,,\ t>0 \qquad (13)\end{aligned}$$

where $\mathbf{E}[\cdot]$ is the expectation operator. Figure 1 shows (a) the conventional WS versus (b) the HWS of $\mathbf{R}_x$ of a windowed $x(t)$. Both show time-varying transfer functions with hyperbolic TF characteristics. The advantage of the HWS in (13), is that it is ideally localized along the three instantaneous frequency curves $f=c_i/t$ in the TF plane. The disadvantage of the conventional WS is that it produces spurious components along hyperbolae since it does not match the intrinsic hyperbolic TF characteristics.

• **Detection problem:** Next, we consider the detection of a known deterministic signal $s(t)$ with hyperbolic TF characteristics in nonstationary Gaussian random noise $n(t)$. Assume that the noise has the correlation function $\mathrm{R}_n(t,\tau)$ whose support region area[3] is less than unity in the hyperbolic SF domain. The test statistic of the optimal likelihood ratio detector is $\mathrm{Re}\{<\mathbf{R}_n^{-1}x,s>\}$ where[4] $\mathbf{R}_n$ is the correlation operator and $x(t)$ is the received signal. Using the *hyperbolic* Weyl correspondence, one obtains $\mathrm{Re}\{<\mathbf{R}_n^{-1}x,s>\}=\iint\mathrm{HWS}_{\mathbf{R}_n^{-1}}(t,f)\mathrm{Re}\{\mathrm{Q}_{xs}(t,f)\}dtdf$ where $\mathrm{Q}_{xs}(t,f)$ is the cross Q-distribution of $x(t)$ and $s(t)$. Similar to the conventional underspread operator approximations in [6, 7], we show that if the hyperbolic SFs of two operators $\mathcal{Y}$ and $\mathcal{S}$ are confined in a small area (jointly underspread), then the hyperbolic WS of the composite operator $\mathcal{YS}$ can be approximated as the product of the hyperbolic WS of each operator [4], i.e.

$$\mathrm{HWS}_{\mathcal{YS}}(t,f)\approx\mathrm{HWS}_{\mathcal{Y}}(t,f)\mathrm{HWS}_{\mathcal{S}}(t,f).$$

For the two correlation operators $\mathbf{R}_n$ and $\mathbf{R}_n^{-1}$, we show that $\mathrm{HWS}_{\mathbf{R}_n\mathbf{R}_n^{-1}}(t,f)\approx\mathrm{HWS}_{\mathbf{R}_n}(t,f)\mathrm{HWS}_{\mathbf{R}_n^{-1}}(t,f)\approx 1$. This simplifies the TF test statistic for detecting a deterministic signal [7]

$$\mathrm{Re}\{<\mathbf{R}_n^{-1}x,s>\}\approx\int_0^\infty\int_{-\infty}^{\infty}\mathrm{Re}\{\mathrm{Q}_{xs}(t,f)\}\Big/\mathrm{HWS}_{\mathbf{R}_n}(t,f)dtdf.$$

[3]The support region of a hyperbolic SF, $\mathrm{HSF}_{\mathrm{R}_n}(\zeta,\beta)$, of the noise process $n(t)$ is the region in (ζ,β) where $\mathrm{HSF}_{\mathrm{R}_n}(\zeta,\beta)\neq 0$.

[4]The inner product is defined as $<x,y>=\int x(t)y^*(t)dt$ and $\mathrm{Re}\{a\}$ is the real part of a.

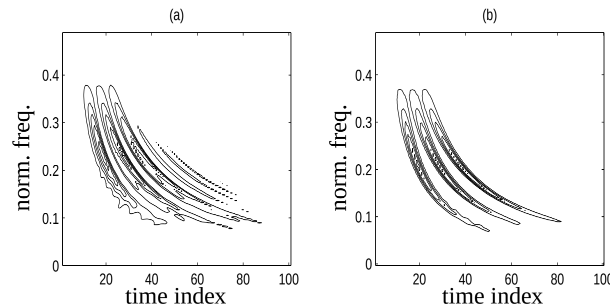

Figure 1. *(a) Weyl symbol, $WS_{\mathbf{R}_x}(t,f)$, and (b) hyperbolic Weyl symbol, $HWS_{\mathbf{R}_x}(t,f)$, of a windowed hyperbolic process $x(t)$.*

6. CONCLUSION

The conventional WS and SF are most useful for systems producing constant time shifts and frequency shifts on the signal. The WS are time-frequency representations that can be interpreted as time-varying spectra for random processes. In this paper, we derived the *new* $\mathrm{P_0WS}$ for systems producing time shifts and scale changes on the signal. Thus, we have extended a useful analysis tool for narrowband systems to wideband systems. Furthermore, using warping techniques, we generalized the conventional narrowband WS and SF and the wideband $\mathrm{P_0WS}$ to *new* WS and SF better matched to dispersive systems. For example, we defined the hyperbolic WS and SF matched to hyperbolic frequency shifts and scale changes, the power WS and SF matched to power law frequency shifts, and the exponential WS and SF matched to exponential frequency shifts. We presented specialized forms of the new WS, SF, and corresponding Weyl correspondences. We also provided application examples in analysis and detection to demonstrate the advantages of our new results.

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