A NUMERICAL ALGORITHM FOR FILTERING AND STATE OBSERVATION

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ABSTRACT

This paper is dealing with a numerical method for data-fitting and estimation of the continuous higher derivatives of a given signal from its non-exact sampled data. The proposed algorithm is a generalization of the algorithm proposed by C. H. Reinsch[1967]. This algorithm is conceived as being a key element in the structure of the numerical observer discussed in our last papers. The presented algorithm seems to be flexible because of the introduction of equivalent conditions of smoothness derived from finite difference methods. Detailed steps of the computational method will be given to evaluate the continuous approximates of higher derivatives of a signal given by its noisy discrete values together with the filtered continuous signal. Satisfactory results have been obtained showing the efficiency of such an algorithm.

Keywords: Spline functions, Numerical differentiation, Observers, Smooth filters.

1. INTRODUCTION

The problem of filtering as well as estimation of higher derivatives of the measurable signals in the presence of noise becomes one of the principal ways to achieve control objectives, construct nonlinear observers and fulfil other physical requirements([1], [2], [8], [11], [12], [13]). This problem has not yet been fully exploited in control and observation theory and necessitates some refinements.

The problem of smoothing and numerical differentiation for non-exact data has received widespread attention in the literature ([3], [4], [6], [7], [10], [14]). Some of these works ([4], [9], [5],[14]) were developed for particular cases. Therefore, the extension to the general case is one of our major interests, and mainly motivates us to improve the quality of observation and filtering with numerical methods.

The main subject of this paper is to introduce a general smoothing algorithm. Detailed steps of the computational method will be given to evaluate the continuous approximates of higher derivatives of a signal given by its noisy discrete values together with the filtered continuous signal. This work is related to the previous work in smoothing data by cubic spline functions developed by C. H. Reinsch, (see [4]). In comparison with the algorithm given by Reinsch, this paper gives a fast solution of the optimization problem with a simple criterion. The solution turns out to be a spline function of arbitrary order, fixed a priori by the user. Higher derivatives are then approximated by differentiating the obtained spline function.

The presented algorithm seems to be flexible because of the introduction of equivalent conditions of smoothness derived from finite difference methods. Moreover, the minimum of the functional to be considered is unique and a fast convergence of Newton methods is expected. We divided our work as follows : The second section is devoted to the formulation of the minimization problem. In Section 3, a detailed solution of the problem is studied. The paper concludes with simulation results and further remarks.

2. PROBLEM FORMULATION

Let (Σ) be a dynamical system with output ζ, and let $(\zeta_1, \cdots, \zeta_n)^t$ be the noisy discrete values which correspond to the equally spaced instants $(t_1, \cdots, t_n)$. One of the famous method to smooth the nonexact data, is to consider the constrained minimization problem

$$\text{minimize} \int_{t_1}^{t_n} \left[\hat{\zeta}^{(m)}\right]^2 \, dt, \tag{1}$$

subject to the constraint

$$\sum_{i=1}^{n} \left[\frac{\hat{\zeta}(t_i) - \zeta(t_i)}{\delta\zeta_i}\right]^2 \leq S, \qquad \hat{\zeta} \in C^{(m)}[t_1,\, t_n]. \tag{2}$$

The notation $\hat{\zeta}^{(m)}$ denotes the m-th derivative of the function $\hat{\zeta}$, $\delta\zeta_i, i = 1, \cdots, n$ are positive numbers taken as estimates of the standard deviation in ζ_i and the number S is redundant used to rescale the quantities $\delta\zeta_i$. In article [4], the author suggests that S could be chosen in the interval $[n-(2n)^{\frac{1}{2}},\, n+(2n)^{\frac{1}{2}}]$. We replace the last constraint by

$$\sum_{i=1}^{n} \left[\hat{\zeta}(t_i) - \zeta(t_i)\right]^2 \leq n\,\sigma^2,$$

if the random noise is supposed to be of zero mean and variance σ^2. Since the vector ζ is available as discrete data, in this article we replace the continuous integral (1) by the following smoothness condition

$$\min \sum_{i=m}^{n-1} \left[\hat{\zeta}_i^{(m)} \, (\Delta t)^m\right]^2. \tag{3}$$

We note $\hat{\zeta}_i^{(m)}$: the finite difference scheme of the m-th derivative of the function $\hat{\zeta}$ at the point i. Δt designates the regular forward difference of t, equal to $t_{i+1} - t_i$. Finally, the problem is formulated as follows

$$\min \sum_{i=m}^{n-1} \left[\hat{\zeta}_i^{(m)} \, (\Delta t)^m \right]^2 .$$

subject to the constraint (2).

3. SOLVING THE OPTIMIZATION PROBLEM BY THE SPLINE FUNCTIONS OF ARBITRARY ORDER

In order to compute the m-th derivative of $\hat{\zeta}$ at point i we will use only the points $\hat{\zeta}_{i-m+1}, \hat{\zeta}_{i-m}, \cdots, \hat{\zeta}_i, \hat{\zeta}_{i+1}$. For example, for $m = 2,\ 3,\ 4$, and 5 the smoothness conditions are

$$\sum_{i=2}^{n-1} \left[\hat{\zeta}_{i-1} - 2\hat{\zeta}_i + \hat{\zeta}_{i+1} \right]^2 ,$$

$$\sum_{i=3}^{n-1} \left[-\hat{\zeta}_{i-2} + 3\hat{\zeta}_{i-1} - 3\hat{\zeta}_i + \hat{\zeta}_{i+1} \right]^2 ,$$

$$\sum_{i=4}^{n-1} \left[\hat{\zeta}_{i-3} - 4\hat{\zeta}_{i-2} + 6\hat{\zeta}_{i-1} - 4\hat{\zeta}_i + \hat{\zeta}_{i+1} \right]^2 ,$$

$$\sum_{i=5}^{n-1} \left[-\hat{\zeta}_{i-4} + 5\hat{\zeta}_{i-3} - 10\hat{\zeta}_{i-2} + 10\hat{\zeta}_{i-1} - 5\hat{\zeta}_i + \hat{\zeta}_{i+1} \right]^2 .$$

respectively. These smoothness conditions are expressed in matrix form as follows

$$\|T\,\hat{\zeta}\|^2 \tag{4}$$

where $\|.\|$ denotes the Euclidean norm and T is as an $(n-m)\times(n)$ matrix of a general row

$$(-1)^{m+j-1} \frac{m!}{(j-1)!(m-j+1)!}, \quad j = 1, \cdots, m+1, \tag{5}$$

and the solution of (1) and (2) turns out to be the minimum of the functional

$$J := \hat{\zeta}^t T^t\, T\hat{\zeta} + \lambda \left\{ (\zeta - \hat{\zeta})^t D^{-2} (\zeta - \hat{\zeta}) + \mu^2 - S \right\}. \tag{6}$$

λ is the Lagrange parameter and μ is an auxiliary variable, $D^{-2} = diag(\delta\zeta_1{}^{-2}, \cdots, \delta\zeta_n{}^{-2})$. We look for the minimum of (6) in the space of the B-spline functions of order $k = 2m$, we replace $\hat{\zeta}$ by

$$\sum_{i=1}^{n} \alpha_i b_{i,2m}(t), \tag{7}$$

such that $\alpha = (\alpha_i,\ i = 1 \cdots, n) \in \mathbb{R}^n$, and $b_{i,2m}$ is the i-th positive B-spline function. We write J in terms of the control vector α as follows

$$\begin{aligned} J := {} & \alpha^t B^t\, T^t\, T\, B\alpha + \\ & \lambda \left\{ (\zeta - B\,\alpha)^t D^{-2} (\zeta - B\,\alpha) + \mu^2 - S \right\}. \end{aligned}$$

with

$$B_{n\times n} = \begin{bmatrix} b_{1,k}(t_1) & b_{2,k}(t_1) & \cdots & b_{n,k}(t_1) \\ b_{1,k}(t_2) & b_{2,k}(t_2) & \cdots & b_{n,k}(t_2) \\ \vdots & \vdots & \ddots & \vdots \\ b_{1,k}(t_n) & b_{2,k}(t_n) & \ldots & b_{n,k}(t_n) \end{bmatrix}.$$

The optimum of the functional $J(\alpha, \mu, \lambda)$ is obtained by differentiating this latter with respect to α, μ and λ. We obtain

$$(T^t\, T + \lambda D^{-2})\, B\,\alpha - \lambda\, D^{-2} \zeta = 0, \tag{8}$$

$$2\mu\lambda = 0, \tag{9}$$

$$(\zeta - B\,\alpha)^t D^{-2} (\zeta - B\,\alpha) + \mu^2 - S = 0. \tag{10}$$

Let u be an $(n-m)\times 1$ vector such that

$$D^2\, T^t\, u = \zeta - B\,\alpha. \tag{11}$$

By substituting (11) in (8), we get

$$(T^t\, T + \lambda D^{-2})(\zeta - D^2\, T^t\, u) = \lambda\, D^{-2} \zeta \tag{12}$$

and after expanding the latter equation, we obtain

$$(T\, D^2\, T^t + \lambda\, I) u = T\,\zeta, \tag{13}$$

where I is an $(n-m)\times(n-m)$ identity matrix. From (13) we write

$$u(\lambda) = (T\, D^2\, T^t + \lambda\, I)^{-1}\, T\,\zeta, \tag{14}$$

and the control vector α is

$$\alpha(\lambda) = B^{-1}\,(\zeta - D^2\, T^t\, u(\lambda)) \tag{15}$$

The Lagrange parameter λ must not be equal to zero. We conclude from (9) that

$$\mu = 0, \tag{16}$$

and

$$(\zeta - B\,\alpha(\lambda))^t D^{-2} (\zeta - B\,\alpha(\lambda)) = S. \tag{17}$$

The unknown Lagrange parameter λ has to satisfy the last equation. Then the control point of the spline will be obtained using equations (14) and (11). Note that

$$\begin{aligned} F^2(\lambda) &:= (\zeta - B\,\alpha)^t D^{-2} (\zeta - B\,\alpha) \\ &= \|D^{-1}(\zeta - B\,\alpha)\|^2 \\ &= \|D\, T^t\, u\|^2. \end{aligned}$$

If we note $Q = D\,T^t$, then the Lagrange parameter is obtained as the solution of the nonlinear equation

$$u^t(\lambda)\, Q^t\, Q\, u(\lambda) = S. \tag{18}$$

By the application of the Newton method, the root λ_r of (17) is obtained after a limited number of the following iterations

$$\lambda_{k+1} = \lambda_k - 2\frac{F^2(\lambda_k)}{\frac{dF^2(\lambda_k)}{d\lambda}} \left[\frac{\sqrt{F^2(\lambda_k)}}{\sqrt{S}} - 1 \right].$$

We have

$$\begin{aligned}\frac{dF^2}{d\lambda} &= 2u^t\,Q^t\,Q\,\frac{du}{d\lambda}\\ &= -2u^t\,Q^t\,Q\,(T\,D^2\,T^t+\lambda\,I)^{-1}u\end{aligned}$$

while $F^2(\lambda_k) > S$, the Newton iteration is

$$\lambda_{k+1} = \lambda_k + \frac{u^t\,Q^t\,Q\,u}{u^t\,Q^t\,Q\,(T\,D^2\,T^t+\lambda_k\,I)^{-1}u} \times \left[\frac{\sqrt{u^t\,Q^t\,Q\,u}}{\sqrt{S}} - 1\right] \qquad (19)$$

Remark 3.1 *The matrix $(T\,D^2\,T^t + \lambda\,I)$ is invertible for any $\lambda \geq 0$.*

Remark 3.2 *The function $F^2(\lambda)$ is strictly decreasing in λ because the matrix $-Q^t\,Q\,(T\,D^2\,T^t + \lambda\,I)^{-1}$ is negative definite for all $\lambda \geq 0$. Consequently, the root of the nonlinear equation $F^2(\lambda) = S$ is unique.*

The Newton iteration involves at each step the calculation of the inverse of the matrix $(T\,D^2\,T^t + \lambda\,I)$. In order to accelerate the rate of convergence of the method, we compute the inverse of the matrix $(T\,D^2\,T^t + \lambda\,I)$ by the use of the Leverrier algorithm. We have

$$(T\,D^2\,T^t+\lambda\,I)^{-1} = \frac{\sum_{i=1}^{n} R_{i-1}\,\lambda^{n-i}}{\sum_{i=0}^{n}\rho_i\,\lambda^{n-i}} \qquad (20)$$

such that

$$\begin{aligned}\rho_i &:= \frac{1}{i}\text{Trace}\left[T\,D^2\,T^t\,R_{i-1}\right], &(21)\\ R_i &:= \rho_i\,I^\star - T\,D^2\,T^t\,R_{i-1}. &(22)\end{aligned}$$

where $I^\star$ is the $(n \times n)$ identity matrix. The matrices $(R_i, i = 0, \cdots, n-1)$ and the coefficients $(\rho_i, i = 0, \cdots, n)$ should be computed before starting the Newton iteration. A fast convergence is expected.

4. THE ALGORITHM

- For a selected order $2m$ and for given breakpoints $(t_1, \cdots, t_n)$, construct an optimal knot sequence $(t)_1{}^n$, (see [3]), and the corresponding B-spline bases $b_{i,2m}$, $i = 1, \cdots, n$.

- Compute the matrix B such that

$$B_{i,j} := b_{j,2m}(t_i),\ i = 1, \cdots, n;\ j = 1, \cdots, n.$$

- Compute the matrix T such that

$$T_{i,j} := \begin{cases} (-1)^{m+j-i}\,C_m{}^{j-i} & \text{for } i = 1, \cdots, n-m; \\ & \text{and } j = i \cdots, m+i, \\ 0 & \text{otherwise.}\end{cases}$$

- Compute the matrix

$$D^{-2} := diag(\delta\zeta_1{}^{-2}, \cdots, \delta\zeta_n{}^{-2}).$$

- Compute the matrices $Q := D\,T^t$ and $(R_i,\ i = 0, \cdots, n-1)$ with the coefficients $(\rho_i,\ i = 0, \cdots, n)$ using eqs (20),(21),(22). If the random noise is of mean zero and variance σ^2, replace the matrix D by the identity matrix and S by $n\sigma^2$.
- Compute the root of the nonlinear equation $F^2(\lambda) = S$ using equations (18), (19), and (20).
- Compute the vector u from (14).
- Solve the linear system

$$B\,\alpha = (\zeta - D^2\,T^t\,u),$$

  with respect to the control points of the spline α. Since the matrix B is positive definite, we write $B = \bar{R}^t\,\bar{R}$: the Cholesky factorization of the matrix B. We have to solve

$$\bar{R}^t\,y = (\zeta - D^2\,T^t\,u)$$

  with respect to y, then

$$\bar{R}\,\alpha = y$$

  with respect to α.
- Compute the derivatives of the spline using the following formulas

$$D^j(\sum_i \alpha_i B_{i,k}) = \sum_i \alpha_i{}^{j+1} B_{i,k-j},$$

  with

$$\alpha_r{}^{j+1} := \begin{cases}\alpha_r & \text{for } j = 0,\\ \frac{1}{k-j}\,\frac{\alpha_r{}^j - \alpha_{r-1}{}^j}{t_{r+k-j}-t_r} & \text{for } j > 0.\end{cases}$$

5. AN EXAMPLE

Here, we consider the system

$$\begin{aligned}\dot{\zeta}_1 &= \zeta_2,\\ \dot{\zeta}_2 &= -150(1+cos(t))\,\zeta_1 - 10(2+sin(t))\,\zeta_2,\\ y &= \zeta_1 + w.\end{aligned}$$

where the scalar output y is supposed to be corrupted by a white noise of zero mean and variance $\sigma^2 = 0.0012$. We consider that the measurements are collected at a regular step $\Delta t = 0.01S$.

6. CONCLUSIONS

Based upon an a priori knowledge of the nature of the noise, the steps of a numerical algorithm used as a filter and an observer were examined. The design problem has been formulated in such a manner that finding the coefficients of the smooth function and its derivatives requires to solve a simple constrained optimization problem. The simplicity of the criterion to be minimized comes from the fact that new conditions of smoothness are proposed. In order to solve the design problem, a resolution of a nonlinear equation and a linear system are required.
Finally, it is possible to extend the idea of the equivalent conditions of smoothness to solve the classical regularization problem discussed in [14]. It is also possible to choose the regularization parameter, in such a manner, that is independent of the statistical properties of noise and only depends on the measurement. This possiblility is currently under investigation and will be reported elsewhere when available.

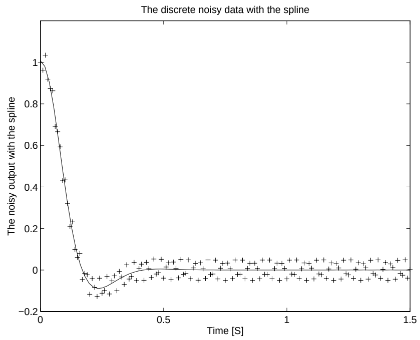

Figure 1: *The filtered output (continuous line), the noisy output (+). In the simulations presented below, the order of the spline is $k = 2m = 6$ and the number of noisy points is $n = 151$. Figure 2 represents the continuous filtered output ζ with the discrete noisy output. In figure 3 and 4 we show the first and the second derivative of the filtered solution with the exact derivatives. We mean by exact derivatives, the solution of the last system without additive noise. Using the last algorithm, we realize that the Newton method converges after 21 iterations, and the Lagrange parameter is, approximately, equal to 0.0207.*

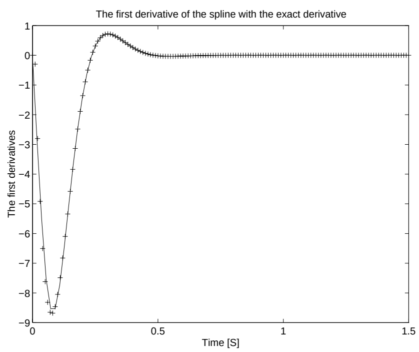

Figure 2: *The exact derivative (+), the derivative of the spline (continuous)*

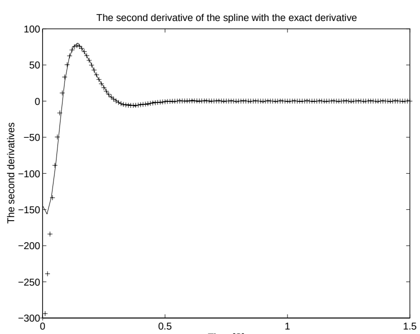

Figure 3: *The exact derivative (+), the derivative of the spline (continuous)*

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