CLOSED-FORM BLIND IDENTIFICATION OF MIMO CHANNELS

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ABSTRACT

We present a closed-form algorithm for blind identification of multiple-input/multiple-output (MIMO) finite-impulse response (FIR) systems driven by digital sources. The algorithm is based on second-order statistics and yields an asymptotically exact estimate of the MIMO channel. We assign distinct spectral signatures to each user through transmitter correlative filters, and exploit this spectral asymmetry to derive the closed-form solution. Simulation results illustrate the good performance of the proposed approach. We compare the mean-square error (MSE) of the MIMO channel estimate against the Cramer-Rao bound, and assess the algorithm capability in rejecting inter-user crosstalk interference.

1. INTRODUCTION

We address the problem of blind identification of digital MIMO-FIR systems. This finds application in Spatial Division Multiple Access (SDMA) architectures for mobile wireless communications. Current approaches [1, 2, 3, 4] use iterative procedures or gradient-based search techniques to solve that problem. Being iterative algorithms, global convergence is not guaranteed *a priori* and several time-consuming restarts may be required. In this paper, we present a *closed-form* blind MIMO system identification algorithm. The main advantage of this algorithm is that it relies on a closed-form solution for the blind MIMO system identification problem. To obtain this closed-form solution, we induce spectral asymmetry between the sources by using adequate correlative filters at each transmitter. This requires no additional power or bandwidth consumption nor synchronization between the users. The correlative-coding approach for MIMO system identification was introduced in [5]. Here, we use a similar framework to propose a new blind identification algorithm which yields asymptotically exact MIMO channel estimates.

The paper is organized as follows. Section 2 introduces the signal model and states the blind identification problem. Section 3 describes the correlative coding approach. An identifiability theorem, which establishes the uniqueness of the channel estimates, is also presented. In Section 4 the closed-form blind MIMO system identification algorithm is derived. Section 5 shows simulation results assessing the performance of our method in the context of the GSM system. Finally, Section 6 presents the conclusions of this work.

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Notation. Matrices (capital) and vectors are in boldface type. $\mathbb{C}^{n\times m}$ is the set of $m \times n$ matrices with complex entries. $\mathcal{R}(\mathbf{A})$ and $\mathcal{N}(\mathbf{A})$ denote, respectively, the range and null-space of matrix $\mathbf{A}$. The notations $(\cdot)^T$, $(\cdot)^*$, $(\cdot)^H$, $(\cdot)^\#$, and $\operatorname{tr}(\cdot)$ stand for transpose, complex conjugate, Hermitian, the Moore-Penrose pseudo-inverse, and the trace operator, respectively. The symbols $\mathbf{I}_n$ and $\mathbf{K}_n$ stand for the $n\times n$ identity and forward-shift (ones in the first lower diagonal) matrices, respectively. Diagonal block matrices with blocks $\mathbf{A}_1, \ldots, \mathbf{A}_n$ are represented by $\operatorname{diag}[\mathbf{A}_1, \ldots, \mathbf{A}_n]$.

2. PROBLEM FORMULATION

Consider a P-input/N-output causal discrete-time noisy linear time-invariant (LTI) digital MIMO system described by the convolution equation

$$\mathbf{x}(k) = \sum_{p=1}^{P} \sum_{l=0}^{L_p-1} \mathbf{h}_p(l)\, s_p(k-l) + \mathbf{w}(k), \tag{1}$$

where $\mathbf{x}(k)$ is an N-dimensional vector of system outputs, $\{\mathbf{h}_p(l) : l = 0, 1, \ldots, L_p - 1\}$ is the finite impulse response (FIR) associated to the pth user's scalar input signal $s_p(k)$, and $\mathbf{w}(k)$ denotes additive noise. Equation (1) can be compactly rewritten as

$$\mathbf{x}(k) = \sum_{p=1}^{P} \mathbf{H}_p \mathbf{s}_p(k) + \mathbf{w}(k) = \mathbf{H}\mathbf{s}(k) + \mathbf{w}(k) \tag{2}$$

where the $N \times L_p$ matrix $\mathbf{H}_p \equiv [\,\mathbf{h}_p(0) \cdots \mathbf{h}_p(L_p-1)\,]$, and $\mathbf{H} \equiv [\,\mathbf{H}_1 \cdots \mathbf{H}_P\,]$ is the $N \times L$ *channel convolution matrix*. $L \equiv L_1 + \cdots + L_P$ is the overall order of the system. The vector $\mathbf{s}(k)$ is obtained by stacking the P vectors $\mathbf{s}_p(k) \equiv [\,s_p(k) \cdots s_p(k-L_p+1)\,]^T$. The input signals $s_p(k)$ are taken from a finite digital modulation-dependent alphabet $\mathcal{A}_p \subset \mathbb{C}$.

In this paper, we study the blind identification of $\mathbf{H}$. We assume that: **(A1)** the number P of users is known and the $N \times L$ channel matrix $\mathbf{H}$ is full-rank with $N \geq L$;

(A2) the sources are uncorrelated and the noise process $\mathbf{w}(k)$ is zero-mean, wide sense stationary, and statistically independent of $\mathbf{s}(k)$, with known autocorrelation matrices $\mathbf{R_W}(l)$.

3. SPECTRAL DIVERSITY

Correlative filters. In most digital communications systems, the emitted source random signals consist of sequences of independent and identically distributed (i.i.d.) symbols from a given constellation set. As a consequence, their respective power spectral densities exhibit a similar flat pattern. In our approach, we propose to color the information sequences prior to transmission, thus inducing a spectral asimmetry between the sources. This will enable a *closed-form* solution for the blind MIMO system identification problem. At the pth source's transmitter, we pass the white sequence generated by the pth user, say $a_p(k) \in \mathcal{A}_p$, through a correlative filter with impulse response

$$c_p(k) = c_p(0)\,\delta(k) + c_p(L_c)\,\delta(k - L_c), \tag{3}$$

and transmit the colored sequence $s_p(k) \equiv (c_p \odot a_p)(k)$. Here, $\delta(k)$ is the discrete-time delta and $\odot$ denotes the convolution operator. For mathematical convenience, the correlation lag L_c in (3) must be greater than or equal to the memory of any FIR channel in the MIMO system, i.e., $L_c \geq \max\{L_1, L_2, \ldots, L_P\}$. Although the delay spreads L_p are unknown *a priori*, in many application scenarios it is possible to adequately overestimate these parameters on the basis of previous field experiments (e.g., in the GSM mobile system, a typical multipath channel profile is available for several environments – hilly,urban,rural [9]). With this correlative pre-processing, the autocorrelation function of the filtered process is

$$r_{s_p}(l) = \eta_p^*\,\delta(l + L_c) + \sigma_p^2\,\delta(l) + \eta_p\,\delta(l - L_c), \tag{4}$$

where $\sigma_p^2 \equiv |c_p(0)|^2 + |c_p(L_c)|^2$ and $\eta_p \equiv c_p(0)^* c_p(L_c)$ denote, respectively, the power and the new correlation peak. In (4), we have assumed white input sequences with unit power. This entails no loss of generality since $\mathbf{H}$ absorbs any multiplicative factor. Consequently, the autocorrelation matrices of the process $\mathbf{s}(k)$ in (2) are, for $l \geq L_c$,

$$\begin{aligned} \mathbf{R_S}(0) &= \operatorname{diag}\left[\sigma_1^2 \mathbf{I}_{L_1}, \cdots, \sigma_P^2 \mathbf{I}_{L_P}\right], \\ \mathbf{R_S}(l) &= \operatorname{diag}\left[\eta_1 \mathbf{K}_{L_1}^{l-L_c}, \cdots, \eta_P \mathbf{K}_{L_P}^{l-L_c}\right]. \end{aligned} \tag{5}$$

For the correlative filters, we adopt the choice $c_p(0) = 1/\sqrt{2}$ and $c_p(L_c) = 1/\sqrt{2} e^{j\,2\pi(p-1)/P}$. This solution distributes the correlation coefficients η_p uniformly around the circle of radius $r = 1/2$ in the complex plane. For completness, we formally state our last assumption: **(A3)** the pth user correlates its zero-mean unit-power symbols $a_p(k) \in \mathcal{A}_p$ so that equations (4) and (5) hold.

System Identifiability. MIMO systems are uniquely defined under the framework established by **(A1)-(A3)**. More precisely, we have the following theorem.

Theorem 1 *Consider the signal model in (2) and suppose that (A1)-(A3) are satisfied. Then, each user convolution matrix $\mathbf{H}_p$ is uniquely determined up to a phase factor by the output autocorrelation matrices $\mathbf{R_X}(0), \mathbf{R_X}(L_c)$, and $\mathbf{R_X}(L_c+1)$. In other words, if $\hat{\mathbf{H}} = \left[\hat{\mathbf{H}}_1 \cdots \hat{\mathbf{H}}_P\right]$ is any channel matrix inducing these same statistics, then*

$$\hat{\mathbf{H}} = \mathbf{H}\boldsymbol{\Theta}, \quad \boldsymbol{\Theta} = \mathit{diag}\left[e^{j\,\theta_1}\mathbf{I}_{L_1}, \cdots e^{j\,\theta_P}\mathbf{I}_{L_P}\right],$$

or, equivalently, $\hat{\mathbf{H}}_p = \mathbf{H}_p e^{j\,\theta_p}$.

A proof of this theorem can be found in [5]. Notice that the residual ambiguity matrix $\boldsymbol{\Theta}$ is *diagonal*, thus effectively decoupling the sources. As it is well known, the remaining phase factors cannot be resolved by just using 2nd order statistics.

4. BLIND IDENTIFICATION ALGORITHM

We present the *closed-form* algorithm that blindly identifies the channel convolution matrix $\mathbf{H}$. The algorithm consists of five main steps.

Step 1: Estimation of L and $\mathbf{H}_0 = \mathbf{H}\mathbf{Q}^H$. The order L is determined as the rank of $\mathbf{R_X}(0) - \mathbf{R_W}(0) = \mathbf{H}\mathbf{H}^H$. In practice, we use the MDL or AIC criteria [10]. Irrespective of the approach chosen, a truncated EVD, $\mathbf{R_X}(0) - \mathbf{R_W}(0) = \mathbf{V}\boldsymbol{\Lambda}^2\mathbf{V}^H$, where $\mathbf{V}$ is a $N \times L$ matrix with orthonormal columns and $\boldsymbol{\Lambda}$ is an invertible diagonal matrix, will be available. It is easy to verify that $\mathbf{H}_0 \equiv \mathbf{V}\boldsymbol{\Lambda}$ satisfies $\mathbf{H}_0 = \mathbf{H}\mathbf{Q}^H$, where $\mathbf{Q} = [\,\mathbf{Q}_1 \cdots \mathbf{Q}_P\,]$ is an $L \times L$ unitary matrix ($\mathbf{Q}_p :\ L \times L_p$).

Step 2: Estimation of $L_1, \ldots, L_P$. Define the matrices $\mathbf{M}_l \equiv \mathbf{H}_0^{\#}(\mathbf{R_X}(l) - \mathbf{R_W}(l))\,\mathbf{H}_0^{\#\,H}$, where $\mathbf{H}_0^{\#} = \boldsymbol{\Lambda}^{-1}\mathbf{V}^H$ is the pseudo-inverse of $\mathbf{H}_0$. Thus, $\mathbf{M}_l = \mathbf{Q}\mathbf{R}_s(l)\,\mathbf{Q}^H$. From the structure of $\mathbf{R_S}(L_c)$ in (5), L_p is the algebraic multiplicity of η_p as an eigenvalue of $\mathbf{M}_{L_c}$. Therefore, if $\mathbf{M}_{L_c} = \mathbf{U}\mathbf{T}\mathbf{U}^H$ is a Schur decomposition ($\mathbf{U}$: unitary, $\mathbf{T}$:upper-triangular), the parameters L_p are obtained by direct inspection of the diagonal of $\mathbf{T}$. From this, the matrix $\mathbf{U}$ is partitioned into P submatrices $\mathbf{U} = [\,\mathbf{U}_1 \cdots \mathbf{U}_P\,]$ ($\mathbf{U}_p :\ L \times L_p$).

Step 3: Estimation of $\boldsymbol{\Pi}_p \equiv \mathbf{Q}_p\mathbf{Q}_p^H$. Since $\mathbf{M}_{L_c}$ is a normal matrix, $\mathbf{T}$ is diagonal [11] (in practice, the off-diagonal entries are negligible). Without loss of generality, let the diagonal of $\mathbf{T}$ be ordered as $\mathbf{R_S}(L_c)$, see (5), then $\mathbf{Q}_p\mathbf{Q}_p^H = \mathbf{U}_p\mathbf{U}_p^H$. That is, the projector associated to the pth user, i.e., the orthogonal projector onto $\mathcal{R}(\mathbf{Q}_p)$, is also available from the Schur decomposition.

Step 4: Estimation of $\mathbf{Q}_p = [\,\mathbf{q}_p(0) \cdots \mathbf{q}_p(L_p - 1)\,]$. Consider the pth user, and set

$$\mathbf{N}_p \equiv \boldsymbol{\Pi}_p\mathbf{M}_{L_c+1} + (\mathbf{I}_L - \boldsymbol{\Pi}_p).$$

Then, $\mathbf{N}_p = \eta_p\mathbf{Q}_p\mathbf{K}_{L_p}\mathbf{Q}_p^H + (\mathbf{I}_L - \boldsymbol{\Pi}_p)$, and $\mathcal{N}\left(\mathbf{N}_p^H\right) = \operatorname{span}\{\mathbf{q}_p(0)\}$. That is, the last right-singular vector of $\mathbf{N}_p^H$ equals $\hat{\mathbf{q}}_p(0) = \mathbf{q}_p(0)e^{j\,\theta_p}$. The remaining columns of $\mathbf{Q}_p$ can be obtained in parallel by setting

$$\hat{\mathbf{q}}_p(l) \equiv \mathbf{M}_l\hat{\mathbf{q}}_p(0) = \mathbf{q}_p(l)\,e^{j\,\theta_p},$$

for $l = 1, \ldots, L_p - 1$. Without loss of generality, we assume in the sequel that $\theta_p = 0$.

Step 5. Estimation of H. Let $\hat{\mathbf{Q}} = \left[\,\hat{\mathbf{Q}}_1 \cdots \hat{\mathbf{Q}}_P\,\right]$. In practice, this matrix is not unitary. To match its structure with this prior knowledge, we project the estimate onto the group of unitary matrices of $\mathbb{C}^{L \times L}$. This can be achieved through a polar decomposition [11], which requires an $L \times L$ SVD. To simplify the notation, we also denote the result of that projection by $\hat{\mathbf{Q}}$. That is, we have an estimate for the channel matrix $\mathbf{Q}$ in the samples $\mathbf{y}(k) \equiv \mathbf{H}_0^{\#}\mathbf{x}(k) = \mathbf{Q}\mathbf{s}(k) + \mathbf{n}(k)$, $(\mathbf{n}(k) \equiv \mathbf{H}_0^{\#}\mathbf{w}(k))$. In Appendix A, we develop an optimal procedure for copying each source's scalar emitted signal $s_p(k)$ from $\mathbf{y}(k)$. The procedure is based on synchronization of the replicas and their coherent recombination to attain high SNR. Let $\hat{s}_p(k)$, $p = 1, \ldots, P$, be the sources' signals estimates obtained by this method, and set $\hat{\mathbf{s}}_p(k) \equiv [\hat{s}_p(k) \cdots \hat{s}_p(k - L_p + 1)]$. That is, we reconstruct the symbols' vector based on their previously known time structure. The MIMO channel convolution matrix is obtained as $\hat{\mathbf{H}} = \mathrm{E}\left\{\mathbf{x}(k)\,\hat{\mathbf{s}}(k)^H\right\}$ or, in a finite K-samples situation, as $\hat{\mathbf{H}} = \mathbf{X}\hat{\mathbf{S}}^{\#}$, where the kth column of $\mathbf{X}$ and $\hat{\mathbf{S}}$ are $\mathbf{x}(k)$ and $\hat{\mathbf{s}}(k)$, respectively.

5. COMPUTER SIMULATIONS

To assess the performance of our algorithm in a multi-user environment, we considered $P = 3$ binary sources, and used typical values of the mobile GSM system. Namely, the standardised impulse response corresponding to the more hostile propagation environment [9] models the air interface channel linking each source to one antenna element. This multipath response consists of six delta impulses modeling six equally-powered independent Rayleigh-fading paths. The delays associated to these paths range from $\tau_{min} = 0\mu s$ to $\tau_{max} = 16\mu s$ in equal steps of $\tau_{step} = 3.2\mu s$. In addition to the amplitude random modulation, each path is also randomly modulated in phase. The sources use the symbol period $T = 3.7\mu s$, and transmit raised-cosine pulses $r_{T/2}(t - T/2)$ with 100% rolloff. With these parameters, each system's impulse response spans over $L_p = 5$ symbol intervals. At the base station, we assumed an antenna array with $D = 6$ sensors and an oversampling factor of $J = 4$. This results in MIMO channel matrices $\mathbf{H}$ of dimension $(DJ) \times (L_1 + L_2 + L_3) = 24 \times 15$. Notice also that each matrix $\mathbf{H}$ contains samples of $M \times P = 18$ GSM impulse responses. The correlative filters were designed as in section 3, with $L_c = 5$. The system's output is corrupted by spatio-temporal white Gaussian noise.

We considered $S = 5$ distinct scenarios $\mathbf{H}$. For each one, the noise variance was chosen to fix the SNR = 20dB, where $\mathrm{SNR} \equiv \mathrm{E}\left\{||\mathbf{H}\mathbf{s}(k)||^2\right\} / \mathrm{E}\left\{||\mathbf{w}(k)||^2\right\}$. The number of data samples used to estimate each $\mathbf{H}$ ranges from $K_{min} = 200$ to $K_{max} = 1000$ in steps of $K_{step} = 100$. For each K, we ran a Monte Carlo simulation consisting of $M = 1000$ independent trials. We computed the corresponding MSE. Figure 1 displays the average results over the $S = 5$ scenarios considered, and against the Cramer-Rao bound (CRB). As we see, the algorithm is asymptotically consistent, meeting in the limit the CRB [6].

We also evaluate the ability of our algorithm to discriminate among the users. Once the channel matrix $\mathbf{H}$ is identified, the pth user's signal is unscrambled from the observations by $\mathbf{y}_p = \mathbf{W}_p^H \mathbf{x}(k)$ (see Appendix A), thus

$$\mathbf{y}_p(k) = \mathbf{W}_p^H \mathbf{H}_p \mathbf{s}_p(k) + \sum_{q \neq p} \mathbf{W}_p^H \mathbf{H}_q \mathbf{s}_q(k) + \mathbf{n}(k). \quad (6)$$

The second term on the right-hand side of (6) measures the residual user's interference crosstalk. Its relative power can be measured by the signal-to-interference ratio (SIR),

$$\mathrm{SIR}_p \equiv \frac{\mathrm{E}\left\{\left|\left|\dot{\mathbf{H}}_p \mathbf{s}_p(k)\right|\right|^2\right\}}{\mathrm{E}\left\{\left|\left|\sum_{q \neq p} \dot{\mathbf{H}}_q \mathbf{s}_q(k)\right|\right|^2\right\}} = \frac{\mathrm{tr}\left(\dot{\mathbf{H}}_p \dot{\mathbf{H}}_p^H\right)}{\sum_{q \neq p} \mathrm{tr}\left(\dot{\mathbf{H}}_q \dot{\mathbf{H}}_q^H\right)},$$

where we have defined $\dot{\mathbf{H}}_q \equiv \mathbf{W}_p^H \mathbf{H}_p$. Figure 2 shows the average results obtained in terms of SIR_p, over the $S = 5$ scenarios simulated. The solid, dashed, and dotted lines refer to the first, second, and third user, respectively. As

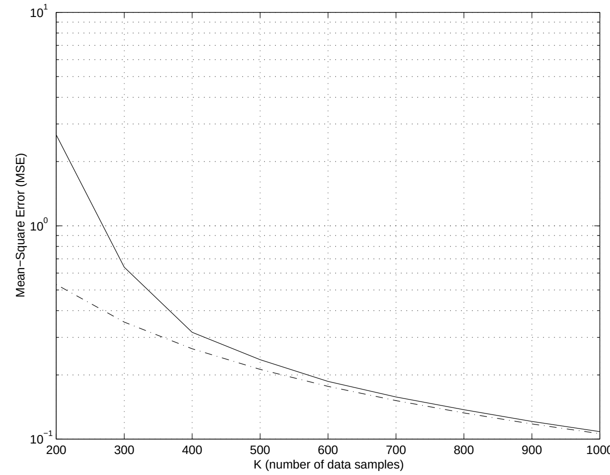

Figure 1: Mean-Square Error of the Algorithm (solid) and CRB (dotted)

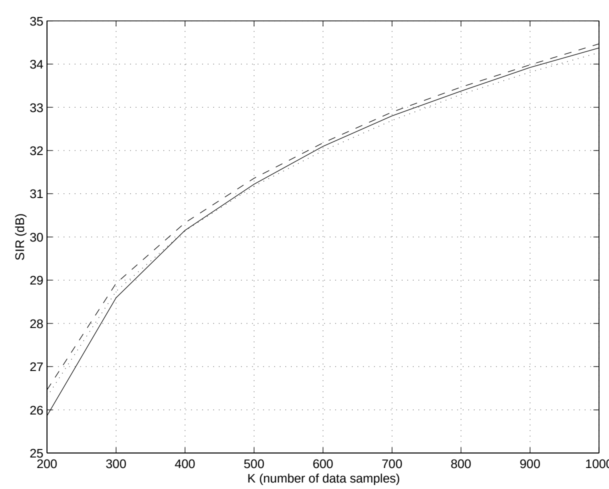

Figure 2: Signal-to-Interference Ratio for User 1,2,3

seen, our method is very effective in rejecting inter-user interference even for small data blocks.

6. CONCLUSIONS

We proposed an asymptotically exact second order-statistics based closed-form algorithm for the blind identification of MIMO-FIR systems driven by digital sources. In our approach, the data streams are adequately colored prior to transmission, in order to assign an unique spectral signature to each user. This requires no additional power, bandwidth, or synchronization between the sources. We presented an identifiability theorem which establishes the uniqueness of MIMO systems under this correlative framework.

Computer simulations illustrated the good performance of the proposed blind identification algorithm, either in terms of mean square-error of the channel estimates or the algorithm's ability in supressing inter-user crosstalk.

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A. SOURCE'S SIGNAL ESTIMATION

Consider the signal model in (2). We develop an optimal estimator for each source's signal $s_p(k)$, given $\mathbf{H}$. Focus on the pth user. The first step consists in isolating its signal from the observations. This is accomplished through an MVDR like beamforming approach, $\mathbf{y}_p(k) \equiv \mathbf{W}_p^H \mathbf{x}(k)$, which preserves the signal of interest and nulls the interference, thus yielding the optimal oblique projector,

$$\mathbf{W}_p = \underset{\mathbf{W}^H \mathbf{H}_q = \mathbf{I}_{L_p}\delta(p-q)}{\arg\min} \mathrm{E}\left\{\left|\left|\mathbf{W}^H \mathbf{x}(k)\right|\right|\right\}^2. \quad (7)$$

After some algebra, we find $\mathbf{W}_p = \Xi_p \mathbf{H}_p \left(\mathbf{H}_p^H \Xi_p \mathbf{H}_p\right)^{-1}$, where $\Xi_p \equiv \mathbf{R}_{\mathbf{W}}^{-1} - \mathbf{R}_{\mathbf{W}}^{-1}\tilde{\mathbf{H}}_p\left(\tilde{\mathbf{H}}_p^H \mathbf{R}_{\mathbf{W}}^{-1}\tilde{\mathbf{H}}_p\right)^{-1}\tilde{\mathbf{H}}_p^H \mathbf{R}_{\mathbf{W}}^{-1}$; here, the matrix $\tilde{\mathbf{H}}_p$ is obtained from $\mathbf{H}$ by deleting the submatrix $\mathbf{H}_p$ and $\mathbf{R}_{\mathbf{W}} \equiv \mathbf{R}_{\mathbf{W}}(0)$. The next step exploits the fact that multiple delayed replicas of $s_p(k)$ are available in the vector $\mathbf{s}_p(k)$. We synchronize the time-delayed signals in the mutichannel vector $\mathbf{y}(k)$, obtaining $\check{\mathbf{y}}_p(k) = \mathbf{1}s_p(k) + \check{\mathbf{n}}(k)$, where $\mathbf{1} = [1 \cdots 1]^T$ (L_p times), and the lth entry of $\check{\mathbf{n}}(k)$ equals the lth entry of $\mathbf{W}_p^H \mathbf{w}(k+l-1)$. The minimum variance unbiased (MVU) estimator for $s_p(k)$ is given by

$$\hat{s}_p(k) = \frac{\mathbf{1}^T \mathbf{R}_{\check{\mathbf{n}}}(0)^{-1}}{\mathbf{1}^T \mathbf{R}_{\check{\mathbf{n}}}(0)^{-1}\mathbf{1}} \check{\mathbf{y}}_p(k).$$

Since $s_p(k)$ belongs to a finite alphabet (notice that this is true even with correlative coding), a minimum-distance receiver is then used to reconstruct the emitted colored symbols. To simplify the notation, we also denote the output of this decison device by $\hat{s}_p(k)$ in the paper (section 4, step 5). Extraction of the white information symbols $a_p(k)$ from $s_p(k)$ is straightforward by a MLSE approach [8].