FREQUENCY WEIGHTED GENERALIZED TOTAL LEAST SQUARES LINEAR PREDICTION FOR FREQUENCY ESTIMATION

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ABSTRACT

This paper presents a frequency weighted generalized total least squares linear prediction for estimating closely spaced sinusoids. In this method, the received data is first processed by a pole-zero prefilter and then a generalized total least squares linear prediction is applied to the prefiltered signal. A procedure of optimizing the generalized solution is introduced. By computer simulations, it is shown that the solution can outperform the existing well known total least squares solutions especially in low signal-to-noise ratio.

I. INTRODUCTION

Resolving closely spaced sinusoids in the presence of noise is a very difficult problem especially when the number of data is small and the signal-to-noise (SNR) is low. In the literature, various methods have been developed in solving this problem [1-3]. Among these methods, eigen-decomposition based techniques such as principle eigenvector (PE) method, minimum norm (MN) method[4], and total least squares (TLS) [5] method have been shown to provide good performance. The MN is proved to be equivalent to the QZTLS [6]. It is well known that the least square prediction error is not a good criterion to provide a robust solution. As a consequence, least square frequency weighted prediction error is used to enhance the linear prediction methods [7].

In [7], the authors have introduced a frequency weighted generalized total least squares (GTLS) method that can generalize the existing TLS methods. The TLS methods in [4] and [5] are the special cases of the GTLS. More importantly, the GTLS can be shown to give better performance.

In this paper, we compare a new pole-zero frequency weighting filter with the commonly used all pole filter [8]. It is shown that the pole-zero filter can give higher SNR than the all-pole filter.

In order to obtain an optimum GTLS solution for any noise condition, an optimum approach is introduced to determine the q-parameter in the GTLS. It is shown that the method can provide satisfactory solutions which can perform better than the TLS under all circumstances.

II. GENERALIZED TLS LINEAR PREDICTION

The model of the received signal y_n is described as:

$$y_n = \sum_{k=1}^{K} a_k \cos(\omega_k n + \theta_k) + w_n \text{, } n = 0,1,\ldots N-1 \qquad (1)$$

where a_k, ω_k and θ_k are the amplitude, frequency and phase of the k-th sinusoid, respectively The noise samples $\{w_n\}$, are assumed to be Gaussian distributed. The frequency estimation problem is to estimate the frequencies and amplitudes of the K sinusoids from the received data record. The following set of linear equation is commonly used to solve this problem.

$$\begin{bmatrix} y_0 & y_1 & \cdots & y_{p-1} \\ y_1 & y_2 & \cdots & y_p \\ \vdots & \vdots & \ddots & \vdots \\ y_{N-p-1} & y_{N-p} & \cdots & y_{N-2} \end{bmatrix} \begin{bmatrix} x_p \\ x_{p-1} \\ \vdots \\ x_1 \end{bmatrix} \approx \begin{bmatrix} y_p \\ y_{p+1} \\ \vdots \\ y_{N-1} \end{bmatrix} \textit{or}$$

$$\mathbf{Ax} \approx \mathbf{b} \qquad (2)$$

where $\mathbf{A}$ is the linear prediction (LP) data matrix, $\mathbf{x}$ is the LP vector and $\mathbf{b}$ is the observation vector. In general the order p of the LP vector $\mathbf{x}$ is larger than K.

Since the data are corrupted by noise, both $\mathbf{A}$ and $\mathbf{b}$ are contaminated and thus TLS approach is more appropriate for solving this LP problem.

The $[\mathbf{A},\mathbf{b}]$ can be written into the following form using singular value decomposition (SVD):

$$[\mathbf{A},\mathbf{b}] = \sum_{i=1}^{p+1} \sigma_i \boldsymbol{u}_i \boldsymbol{v}_i^{+} \qquad (3)$$

where $\boldsymbol{u}_i$ is a $N \times 1$ vector and $\boldsymbol{v}_i$ is a $(p+1)\times 1$ vector, and $^+$ denotes the complex conjugate matrix transpose. The principal singular vectors approximation of $[\mathbf{A},\mathbf{b}]$ using the largest M singular values is described by

$$[\hat{\mathbf{A}}, \hat{\mathbf{b}}] = \sum_{i=1}^{M} \sigma_i \boldsymbol{u}_i \boldsymbol{v}_i^+ \tag{4}$$

Then the TLS solution for $[\hat{\mathbf{A}}, \hat{\mathbf{b}}]$ is spanned by the set of singular vectors $[\boldsymbol{v}_{M+1}, \cdots, \boldsymbol{v}_{p+1}]$, Mathematically, the TLS solution is given by

$$\boldsymbol{x}_{GTLS} = -\sum_{i=M+1}^{p+1} \frac{\beta_i v_{i,p+1}}{\sum_{k=M+1}^{p+1} \beta_k |v_{k,p+1}|^2} \boldsymbol{v}_i^{'} \tag{5}$$

where $\boldsymbol{v}_i^{'}$ is the vector of first p elements of $\boldsymbol{v}_i$ and $\{\beta_i\}$ are some positive real numbers.

Let β_i be related to the singular value σ_i as follow:

$$\beta_i = \sigma_i^{-q} \tag{6}$$

Then, the general TLS solution is given by

$$\boldsymbol{x}_{GTLS} = -\sum_{i=M+1}^{p+1} \frac{\sigma_i^{-q} v_{i,p+1}}{\sum_{k=M+1}^{p+1} \sigma_k^{-q} |v_{k,p+1}|^2} \boldsymbol{v}_i^{'} \tag{7}$$

When $q \to \infty$, $\boldsymbol{x}_{GTLS}$ becomes $\boldsymbol{x}_{MN}$; while $q = 0$, $\boldsymbol{x}_{GTLS} = \boldsymbol{x}_{QZTLS}$, the solution given in [5].

According to GTLS solution in (7) the prediction error $\boldsymbol{e}$ is given by

$$\boldsymbol{e} = \left[\sum_{i=M+1}^{p+1} \sigma_i \boldsymbol{u}_i \boldsymbol{v}_i^+ \right] \begin{bmatrix} \boldsymbol{x}_{GTLS} \\ -1 \end{bmatrix} \tag{8}$$

The mean squared prediction error is therefore given by

$$E_{GTLS} = \boldsymbol{e}^T \boldsymbol{e} = \frac{\sum_{i=M+1}^{p+1} \sigma_i^{2(1-q)} |v_{i,p+1}|^2}{\left[\sum_{i=M+1}^{p+1} \sigma_i^{-q} |v_{i,p+1}|^2 \right]^2} \quad . \tag{9}$$

In the next section, we discuss about a new pole-zero frequency weighting filter to improve the signal-to-noise ratio for the GTLS linear prediction.

III. POLE-ZERO FREQUENCY WEIGHTING FILTER

In the literature, it has been shown that an all-pole filter as obtained from the linear prediction can be set as a prefilter for the data to effectively enhance the SNR in the signal band. In this section, we discuss about the use of a pole-zero filter as the prefilter and it is shown that the new filter structure can provide better SNR than the all-pole filter.

The transfer function of the pole-zero filter is given by

$$W(z) = 1 - \sum_{i=1}^{p} a_{LS}(i) \gamma^i z^{-i} \Big/ 1 - \sum_{i=1}^{p} a_{LS}(i) z^{-i} \tag{10}$$

where $\{a_{LS}(i)\}_{i=1,..,p}$ is the linear prediction coefficients obtained by least squares method. Unlike the all pole filter whose poles locations are varied according to the noise condition, the zero of the pole-zero filter are controlled by a parameter γ.

In order to compare the two filter structures, we consider a single sinusoid plus Gaussian noise. Assuming that the linear predictor obtained by least squares method is given by

$$A_\gamma(z) = 1 - 2\gamma\rho \cos\omega_0 z^{-1} + \gamma^2 \rho^2 z^{-2} \tag{11}$$

where ω_0 is the frequency of the sinusoid, ρ is the radius of the roots of the predictor and γ is a controlling parameter. Let a_{AP} and a_{PZ} denote the amplitudes of the sinusoid at the output of the all-pole filter and the pole-zero filter, respectively. It is shown that

$$a_{AP}^2 = \left| \frac{a_0}{A_\gamma(e^{j\omega_0})} \right|^2$$

$$= \frac{a_0^2}{1 + \gamma^4\rho^4 + 4\gamma^2\rho^2 \cos^2\omega_0 + 2\gamma^2\rho^2 \cos^2 2\omega_0 - 4\gamma\rho(1 + \gamma^2\rho^2)\cos^2\omega_0}$$

$$a_{PZ}^2 = a_0^2 \left| \frac{A_\gamma(e^{j\omega_0})}{A_1(e^{j\omega_0})} \right|^2$$

$$= a_0^2 \frac{1 + \gamma^4\rho^4 + 4\gamma^2\rho^2 \cos^2\omega_0 + 2\gamma^2\rho^2 \cos 2\omega_0 - 4\gamma\rho(1 + \gamma^2\rho^2)\cos^2\omega_0}{1 + 4\rho^2 \cos^2\omega_0 + \rho^4 - 4\rho\cos^2\omega_0 + 2\rho^2 \cos 2\omega_0 - 4\rho^3 \cos^2\omega_0}$$

where a_0 is the amplitude of the sinusoid in the signal model. Similarly, define σ_{AP}^2 and σ_{PZ}^2 as the output noise power of the all-pole filter and the pole-zero filter, respectively. We obtain

$$\sigma_{AP}^2 = \sigma_w^2 \left\{ 1 + \frac{\gamma^2\rho^2 \left[(1 + \gamma^2\rho^2)(4\cos^2\omega_0 + \gamma^2\rho^2) - 8\gamma^2\rho^2 \cos^2\omega_0 \right]}{(1 - \gamma^2\rho^2)(1 + \gamma^4\rho^4 - 2\gamma^2\rho^2 \cos 2\omega_0)} \right\}$$

$$\sigma_{PZ}^2 = \sigma_w^2 \left\{ 1 + \frac{(1 + \rho^2)\left[4(1-\gamma)^2 \rho^2 \cos^2\omega_0 + \rho^4 (1 - \gamma^2)^2 \right] - 8\rho^4 (1 - \gamma^2)(1 - \gamma)\cos^2\omega_0}{(1 - \rho^2)(1 + \rho^4 - 2\rho^2 \cos 2\omega_0)} \right\}$$

where σ_w^2 is the noise power in the signal model.

In Fig.1, we plot the amplitude gain versus noise power by varying the controlling parameter γ from 0 to 1 for both filters. It is shown that at the same amplitude gain, the pole-zero filter has much lower noise power.

By computer simulations, the value of the controlling parameter close to zero can provide good performance for low SNRs while for high SNRs, the value of γ above 0.6 will give better result. Generally in the range between 0.2

to 0.6 can provide a satisfactory result over a wide range of input SNRs.

IV. OPTIMUM GTLS SOLUTION

The GTLS provides a general form of total least squares solution in terms of a parameter q. In the following, we discuss about using mean squared frequency weighted error (MSWE) as a criterion to determine the value of q in the GTLS. According to the analysis in section II, the MSWE is given by equation (9) with the use of a prefilter. The mean squared frequency error (MSFE) is defined as

$$MSFE = 10\log_{10} \frac{1}{KL}\sum_{l=1}^{L}\sum_{i=1}^{K}(\hat{\omega}_i^{(l)} - \omega_i)^2 \quad (12)$$

where L equals the number of trials and $\hat{\omega}_i^{(l)}$ is the estimate of the i-th tone frequency. In the next section, an experiment of two closely spaced sinusoids is carried out for illustrating the performance of the GTLS. The contour plots of the MSWE and the MFSE of the GTLS for SNR=5dB are shown in Fig 2a and 2b, respectively. It is observed that the optimum solutions of two criteria are located quite close to each other alone the q-axis. As a result, it shows that the MSWE is a possible criterion for determining the value of q. According to this approach, the GTLS with optimum q is a solution in the total least squares domain that has the minimum mean squared weighted error.

As shown in Fig. 2, the error surface is a simple quadratic function in the vicinity of q=0. We can use a first order approximation of the GTLS solution to determine an optimum q. Accordingly, the GTLS solution, $\boldsymbol{x}_{GTLS}(q)$, can be written as

$$\boldsymbol{x}_{GTLS}(q) = \boldsymbol{x}_{GTLS}(0) + q\boldsymbol{x}_{GTLS}^{(1)}(0) \quad (13)$$

where $\boldsymbol{x}_{GTLS}^{(1)}(0)$ is the first derivative of $\boldsymbol{x}_{GTLS}(q)$ at q=0, given by

$$\boldsymbol{x}_{GTLS}^{(1)}(0) = \begin{bmatrix} \boldsymbol{x}^{(1)} \\ 0 \end{bmatrix}$$

where

$$\boldsymbol{x}^{(1)} = -\sum_{i=M+1}^{p+1} \frac{v_{i,p+1}\left(\sum_{k=M+1}^{p+1}\log\sigma_k v_{k,p+1}^2 - \log\sigma_i \sum_{k=M+1}^{p+1} v_{k,p+1}^2\right)}{\left[\sum_{k=M+1}^{p+1} v_{k,p+1}^2\right]^2}$$

It is noted that $\boldsymbol{x}_{GTLS}(0)$ in (13) is the QZTLS solution. Therefore, the $\boldsymbol{x}_{GTLS}(q)$ together with an optimum q can provide a lower MSWE than that of the TLS solution.

Let $\boldsymbol{e}(q)$ denote the weighted prediction error of $\boldsymbol{x}_{GTLS}(q)$ as given by (8). It can be written as

$$\begin{aligned}\boldsymbol{e}(q) &= \boldsymbol{e}(0) + q\left(\sum_{i=M+1}^{p+1}\sigma_i \boldsymbol{u}_i \boldsymbol{v}_i^{+}\right)\begin{pmatrix} \boldsymbol{x}^{(1)} \\ 0 \end{pmatrix} \\ &= \boldsymbol{e}(0) + q\boldsymbol{e}^{(1)}(0)\end{aligned} \quad (14)$$

By setting $\frac{\partial}{\partial q}\left(\boldsymbol{e}(q)^T \boldsymbol{e}(q)\right) = 0$, the optimal q is given by,

$$q_{opt} = -\frac{\boldsymbol{e}(0)^T \boldsymbol{e}^{(1)}(0)}{\boldsymbol{e}^{(1)}(0)^T \boldsymbol{e}^{(1)}(0)} \quad (15)$$

This solution can be shown to provide significant performance gain over the QZTLS especially for low SNRs.

V. SIMULATION RESULTS

The signal model of the experiments is defined in (1). We consider two closely spaced sinusoids with frequencies ω_1=0.7813π and ω_2=0.7617π and amplitudes equal. The phases of the sinusoids are randomly selected from (0,2π). The noise $\{w_n\}$ is Gaussian distributed. The length N is set equal to 128 and the number of principal eigenvectors, M, is set equal to 4. The results are averaged over 100 trials.

To illustrate the performance of the GTLS for using the two types of prefilters, the mean squared frequency errors of the GTLS with q=-5,0,20,∞ and q_{opt} are summarized in Table 1. It is worth to mention that the GTLS using q_{opt} means that the optimum q is computed according to (15) for each trial. In the experiments, the parameter γ in the pole-zero filter, is set equal to 0.3 The results show that the GTLS with q_{opt} gives the best performance for all SNRs. It shows that the optimization procedure can give a robust solution under any SNR.

CONCLUSIONS

An optimum generalized total least squares solution is presented in this paper. A procedure of computing the optimum q parameter in the GTLS is introduced and it is shown that the optimum solution can outperform the existing TLS solutions. Further, the new pole-zero prefilter is shown to perform better than the commonly used all pole filter. The GTLS together with the pole-zero prefilter provide a robust method to estimate closely spaced sinusoids in low SNR.

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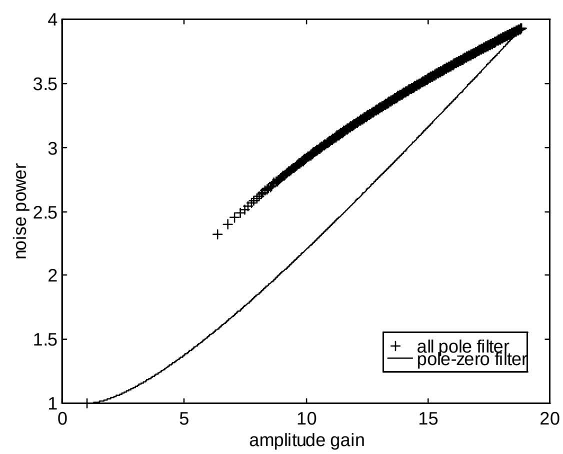
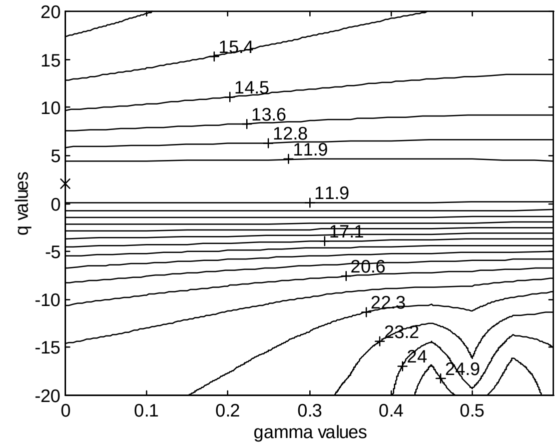


Fig 1 Amplitude gain versus noise power



(a)