INTERPOLATION OF NONSTATIONARY FIELDS WITH STATIONARY INCREMENTS

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ABSTRACT

The problem of the linear interpolation of nonstationary multidimensional processes with stationary increments is studied. The expressions of the interpolation filters and of the estimation error are derived, which generalize the results of the interpolation theory for stationary processes. Both finite and infinite extent interpolation are considered. An application to the interpolation of an underwater depth map is presented.

1. INTRODUCTION

Linear mean-square interpolation of wide-sense stationary processes is a classical problem. When the number of samples is finite, the solution is obtained by solving the so-called normal equations (see [2] for a survey and recent contributions on this subject). When the number of samples is infinite, the solution is given by a simple expression in the frequency domain. Nonstationary processes do not enter in this framework and their statistical interpolation is a more difficult task. In this paper, we restrict our study to nonstationary processes with stationary increments [5], which are valuable models for many natural phenomena in physics, hydrology, economics, turbulence and communications. Among the most famous processes in this class are the fractional Brownian motions [3], for which approximate interpolation methods based on midpoint displacement have been developped [7]. In the general case of nonstationary fields with stationary increments, we will introduce the so-called structure function [1], which characterizes the second-order statistical properties of these fields and will be used to model them. By reconsidering the problem of finite extent interpolation within this framework, we show that the solution is obtained by solving an unconstrained linear system and its expression, as well as the mean-square estimation error, are given in terms of the structure function. In the infinite extent case, we show that the solution takes a simple form by using the spectral density of the increments of the field. An application to the interpolation of an underwater depth map of the Var Canyon (France) is presented to illustrate these results.

The paper is organized as follows: in Section 2 we introduce the model of nonstationary processes with stationary increments. Section 3 presents the interpolation methods in the cases of finite and infinite extent. In Section 4, we present an application and conclude the paper.

2. NONSTATIONARY PROCESSES WITH STATIONARY INCREMENTS

We first introduce the model of nonstationary signals with stationary increments which is used in this work. With little loss of generality, we shall assume throughout this paper that all the considered processes are zero-mean.

A nonstationary real process $\{F(\mathbf{x}),\ \mathbf{x} \in \mathbb{S}\}$ with $\mathbb{S} = \mathbb{Z}^n$ or $\mathbb{S} = \mathbb{R}^n$ has stationary increments if the autocorrelation function of its increments:

$$\mathrm{E}\left\{(F(\mathbf{x}) - F(\mathbf{x}-\Delta))\ (F(\mathbf{x}') - F(\mathbf{x}'-\Delta))\right\}$$

only depends on the relative distance between $\mathbf{x}$ and $\mathbf{x}'$ and on the value of the increments Δ. One can easily show [4] that this is equivalent to the fact that the variance of the increments does not depend on the origin. In other words, there exists a function, which will be called the structure function φ_F, such that $\varphi_F(\Delta) = \mathrm{var}\left\{F(\mathbf{x}) - F(\mathbf{x}-\Delta)\right\}$. The correlation function of the process can be expressed as:

$$\begin{aligned}&\mathrm{E}\left\{F(\mathbf{x})F(\mathbf{x}')\right\} \\ &= \frac{1}{2}\left[\varphi_F(\mathbf{x}) + \varphi_F(\mathbf{x}') - \varphi_F(\mathbf{x}-\mathbf{x}')\right] + \mathrm{E}\left\{F^2(0)\right\},\end{aligned}$$

which implies that the structure function and $\mathrm{E}\left\{F^2(0)\right\}$ completely characterizes a zero-mean Gaussian process. Note that a stationary process is a special case of a process with stationary increments as defined above. As can be expected, the structure function of such a process is closely related to its autocorrelation function $\Gamma(\Delta) = \{F(\mathbf{x})F(\mathbf{x}-\Delta)\}$, as it is straightforward to check that $\varphi(\Delta) = 2\ (\Gamma(\mathbf{0}) - \Gamma(\Delta))$. We can therefore introduce, as for the correlation function, a biased empirical estimator of the structure function of a nonstationary process:

$$\widehat{\varphi}_F(\Delta) = \frac{1}{N}\sum_{\mathbf{x}\in\mathcal{D}(\Delta)}\left[F(\mathbf{x}) - F(\mathbf{x}-\Delta)\right]^2, \qquad (1)$$

where N is the size of a finite sample field $\mathcal{D}$ and $\mathcal{D}(\Delta) = \{\mathbf{x} \in \mathcal{D} \mid \mathbf{x} - \Delta \in \mathcal{D}\}$. We can also construct an unbiased empirical estimator by normalizing by the effective number of points in $\mathcal{D}(\Delta)$.

The fractional Brownian motion [6], is obtained for $\varphi_F(\Delta) = ||\Delta||^{2H}$, where H is the Hurst parameter of the process. It is an isotropic process. Remark that, in this case, the Fourier transform of the structure function exists only in a distributional sense [4]. The introduction of the structure function allows us to define anisotropic processes with stationary increments.

In the sequel, we will consider discrete time/space signals $\{F(\mathbf{k}),\ \mathbf{k} \in \mathbb{Z}^n\}$.

3. INTERPOLATION METHOD

3.1. Finite extent interpolation

In this section, we realize the linear interpolation of the signal based on a finite number of samples, which may be localized on a regular or irregular grid.

Let $F(\mathbf{n})$ be the value to be estimated and let $\mathcal{S}$ be a finite subset of $\mathbb{Z}^n \setminus \{\mathbf{0}\}$ which defines a finite neighbourhood $\{\mathbf{n} - \mathbf{p},\ \mathbf{p} \in \mathcal{S}\}$ of the point $\mathbf{n}$. Remark that we do not impose any constraint on the neihgbourhood, so that it can be symmetric or not. The estimated value $\widehat{F}(\mathbf{n})$ will be:

$$\widehat{F}(\mathbf{n}) = \sum_{\mathbf{p} \in \mathcal{S}} h_{\mathbf{n}}(\mathbf{p}) F(\mathbf{n} - \mathbf{p}), \tag{2}$$

where $h_{\mathbf{n}}(\mathbf{p})$ are the coefficients of the filter to be optimized. It can be easily shown that this filter is shift-invariant (i.e. its coefficients do not depend on the position $\mathbf{n}$ of the estimated sample) if $\sum_{\mathbf{p} \in \mathcal{S}} h_{\mathbf{n}}(\mathbf{p}) = 1$. We can then denote the coefficients by $h(\mathbf{p})$. For an arbitrary position $\mathbf{p}_0 \in \mathcal{S}$, we have: $h(\mathbf{p}_0) = 1 - \sum_{\mathbf{p} \in \mathcal{S}_0} h(\mathbf{p})$, where $\mathcal{S}_0 = \mathcal{S} \setminus \{\mathbf{p}_0\}$. The interpolation coefficients are estimated by minimizing the mean square estimation error:

$$\varepsilon^2 = \mathrm{E}\left\{ \left[F(\mathbf{p}_0) - \sum_{\mathbf{p} \in \mathcal{S}_0} h(\mathbf{p}) \left(F(\mathbf{p}_0) - F(\mathbf{p}) \right) \right]^2 \right\}.$$

The problem therefore reduces to the linear mean square estimation of $F(\mathbf{p}_0)$ from $F(\mathbf{p}_0) - F(\mathbf{p}),\ \mathbf{p} \in \mathcal{S}_0$. The normal equations can be written, by using the structure function φ_F of the process F: $\forall \mathbf{k} \in \mathcal{S}_0$,

$$\begin{aligned} &\sum_{\mathbf{p} \in \mathcal{S}_0} \left[\varphi_F(\mathbf{p}_0 - \mathbf{k}) + \varphi_F(\mathbf{p} - \mathbf{p}_0) - \varphi_F(\mathbf{p} - \mathbf{k}) \right] h(\mathbf{p}) \\ &\qquad = \varphi_F(\mathbf{p}_0 - \mathbf{k}) + \varphi_F(\mathbf{p}_0) - \varphi_F(\mathbf{k}). \end{aligned} \tag{3}$$

The resulting mean-square error is

$$\begin{aligned} \varepsilon^2 = \sum_{\mathbf{p} \in \mathcal{S}} h_{\mathbf{p}} \mathrm{E}\left\{ F(\mathbf{p}_0) F(\mathbf{p}) \right\} = \frac{1}{2} \Big\{ \varphi_F(\mathbf{p}_0) \\ + \sum_{\mathbf{p} \in \mathcal{S}} h(\mathbf{p}) \varphi_F(\mathbf{p}) - \sum_{\mathbf{p} \in \mathcal{S}} h(\mathbf{p}) \varphi_F(\mathbf{p} - \mathbf{p}_0) \Big\}. \end{aligned}$$

and it can be checked that it does not depend on $\mathbf{p}_0$.

3.2. Infinite extent interpolation

In this section, we study the case of the estimation of the missing 1D or 2D data from an infinite number of observations. We suppose data spaced on a regular grid and we are looking for the values of the process corresponding to a finer sampling.

3.2.1. 1D case

The values at missing points can be estimated by:

$$\widehat{F}(nM + p) = \sum_{k=-L}^{L} h(kM) F((n-k)M),$$

where $p \in \{1, \ldots, M-1\}$. Let us denote the unknown values $F(nM)$ by $A(n)$. The condition to have a shift-invariant interpolation filter is, as previously, $\sum_{k=-L}^{L} h(kM) = 1$. Then, it can be easily shown that the interpolation of the nonstationary process with stationary increments reduces to a linear filtering of its increments $\Delta A(n) = A(n) - A(n-1)$:

$$\widehat{F}(nM + p) = A(n) + \sum_{k=-L}^{L} g(k) \Delta A(n-k),$$

where g is the impulse response of a filter to be optimized in the mean-square sense. Let us now assume that $L \to \infty$. By denoting $B(n) = F(nM+p) - F(nM)$, the frequency response of the filter is given by:

$$G(\omega) = \frac{\mathcal{S}_{B\,\Delta A}(\omega)}{\mathcal{S}_{\Delta A}(\omega)},$$

where $\mathcal{S}_{\Delta A}(\omega)$ is the spectral density of the increment process and $\mathcal{S}_{B\,\Delta A}(\omega)$ is the inter-spectral density of the processes B and ΔA. One can show that these spectral densities are related to the structure function φ_F as follows:

$$\begin{aligned} \mathcal{S}_{\Delta A}(\omega) &= -\frac{1}{2} \left| 1 - e^{j\omega} \right|^2 \widehat{\varphi}_A(\omega), \\ \widehat{\varphi}_A(\omega) &= \frac{1}{M} \sum_{k=0}^{M-1} \widehat{\varphi}_F \left(\frac{\omega + 2k\pi}{M} \right) \end{aligned} \tag{4}$$

and

$$\mathcal{S}_{B\,\Delta A}(\omega) = -\frac{1}{2M}\sum_{k=0}^{M-1}\left(e^{jp\frac{\omega+2k\pi}{M}}-1\right) \times \left(1-e^{j\omega}\right)\widehat{\varphi}_F\left(\frac{\omega+2k\pi}{M}\right).$$

Note that even when $\widehat{\varphi}_F(\omega)$ exists only in a distributional sense, the expression $\left(1-e^{j\omega}\right)\widehat{\varphi}_F(\omega)$ corresponds to a well-defined function.

With these results, the mean-square interpolation error is:

$$\varepsilon^2 = \varphi(p) - \frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{\left|\mathcal{S}_{B\,\Delta A}(\omega)\right|^2}{\mathcal{S}_{\Delta A}(\omega)}d\omega.$$

3.2.2. 2D case

The interpolated values are now given by:

$$\widehat{F}(nM+p, mM+q) = \sum_{k=-L}^{L}\sum_{l=-L}^{L} h(kM, lM)F((n-k)M, (m-l)M)$$

where $(p,q) \in \{1,\ldots,M-1\}^2$. As previously, $F(nM, mM)$ is designated by $A(n,m)$.

Let us denote the increments of the process in the x (resp. y) direction by $\Delta A^{1,0}(n,m) = A(n,m) - A(n-1,m)$ (resp. $\Delta A^{0,1}(n,m) = A(n,m) - A(n,m-1)$). We also define the increment of order $(1,1)$ by

$$\Delta A^{1,1}(n,m) = A(n,m) - A(n-1,m) - A(n,m-1) + A(n-1,m-1).$$

As the field $F(n,m)$ has stationary increments, all the above three processes are stationary. Moreover, one can show that the linear interpolation of the nonstationary process $F(n,m)$ can be expressed as a linear combination of $A(n,m)$ and its increments of order $(1,0)$, $(0,1)$ and $(1,1)$. Let us now assume that $L \to \infty$. Since the increment of order $(1,1)$ can be written as a linear combination of the increments of order $(1,0)$ or $(0,1)$:

$$\begin{aligned}\Delta A^{1,1}(n,m) &= \Delta A^{1,0}(n,m) - \Delta A^{1,0}(n,m-1)\\ &= \Delta A^{0,1}(n,m) - \Delta A^{0,1}(n-1,m),\end{aligned}$$

the interpolated value at the position $(nM+p, mM+q)$ takes the following form:

$$\begin{aligned}\widehat{F}(nM+p, mM+q) &= A(n,m)\\ &+ \sum_{k=-\infty}^{\infty}\sum_{l=-\infty}^{\infty} g_{k,l}^{1,0}\Delta A^{1,0}(n-k, m-l)\\ &+ \sum_{k=-\infty}^{\infty}\sum_{l=-\infty}^{\infty} g_{k,l}^{0,1}\Delta A^{0,1}(n-k, m-l),\end{aligned}$$

where $g_{k,l}^{1,0},\ g_{k,l}^{0,1}$ are the impulse responses of linear filters to be optimized.

The minimization of the mean square error ε^2 corresponds to the linear estimation of $B(n,m) = F(nM+p, mM+q) - F(nM, mM)$ from the increments $\Delta A^{1,0}(n-k, m-l)$ and $\Delta A^{0,1}(n-k, m-l)$. The spectral and inter-spectral densities of these fields can easily be deduced from their correlation functions and one can verify that

$$\mathcal{S}_{\Delta A^{1,0}}(\omega_x,\omega_y)\,\mathcal{S}_{\Delta A^{0,1}}(\omega_x,\omega_y) = \left|\mathcal{S}_{\Delta A^{1,0}\Delta A^{0,1}}(\omega_x,\omega_y)\right|^2,$$

which means that the estimation problem has an infinite number of solutions. (This result is not surprising, since there are several ways of reconstructing $A(n-p, m-q)$ from $A(n,m)$ and its increments of order $(1,0)$ and $(0,1)$.) The frequency response of the filters can be chosen as follows:

$$\begin{aligned}G_{\Delta A^{1,0}}(\omega_x,\omega_y) &= \frac{\mathcal{S}_{B\,\Delta A^{1,0}}(\omega_x,\omega_y)}{\mathcal{S}_{\Delta A^{1,0}}(\omega_x,\omega_y)}\\ G_{\Delta A^{0,1}}(\omega_x,\omega_y) &= 0.\end{aligned}$$

The mean-square error reads then

$$\varepsilon^2 = \varphi(p,q) - \frac{1}{4\pi^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\frac{\left|\mathcal{S}_{B\,\Delta A^{1,0}}(\omega_x,\omega_y)\right|^2}{\mathcal{S}_{\Delta A^{1,0}}(\omega_x,\omega_y)}d\omega_x d\omega_y,$$

where

$$\begin{aligned}&\mathcal{S}_{B\,\Delta A^{1,0}}(\omega_x,\omega_y)\\ &\quad= -\frac{1}{2M^2}\sum_{k=0}^{M-1}\sum_{l=0}^{M-1}\left(e^{jp\frac{\omega_x+2k\pi}{M}}e^{jq\frac{\omega_y+2l\pi}{M}}-1\right)\\ &\quad\times\left(1-e^{j\omega_x}\right)\widehat{\varphi}_F\left(\frac{\omega_x+2k\pi}{M},\frac{\omega_y+2l\pi}{M}\right)\\ &\mathcal{S}_{\Delta A^{1,0}}(\omega_x,\omega_y) = -\frac{1}{2}\left|1-e^{-j\omega_x}\right|^2\widehat{\varphi}_A(\omega_x,\omega_y),\end{aligned}$$

where $\widehat{\varphi}_A(\omega_x,\omega_y)$ is the Fourier transform of the structure function of the field $A(n,m)$. This Fourier transform is related to $\widehat{\varphi}_F(\omega_x,\omega_y)$ by a formula similar to (4).

4. APPLICATION TO THE SEA-BED MODELING

As an application, we have considered an underwater depth map (271×271 points) of the Var Canyon (France) (see Fig. 1). The structure function was estimated from real data, by using Equation (1). It is represented in Fig. 2. In order to model and interpolate this map, we have chosen the following parametric model for the structure function:

$$\varphi(\rho\cos\theta, \rho\sin\theta) = a_0\left(\frac{\rho}{R}\right)^{\beta_0}\left|\cos(\theta-\theta_0)\right|^{\gamma_0} + a_1\left(\frac{\rho}{R}\right)^{\beta_1}\left|\sin(\theta-\theta_0)\right|^{\gamma_1},\ \rho\geq 0,$$

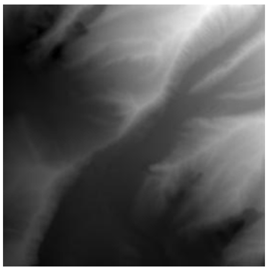

Figure 1: Original data of the Var Canyon.

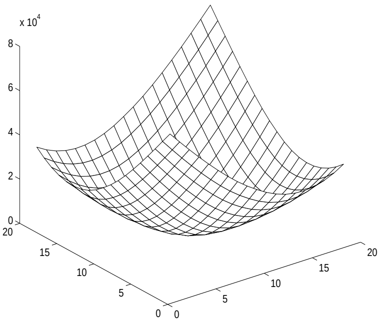

Figure 2: Structure function estimated from the real data.

where R is the interpolation step, here equal to 4 (three new points were intercalated between any two points on the original grid). This model was fitted to the structure function estimated from the real data and the parameters have been computed by a mean-square optimization technique. The resulting values of the parameters are: $a_0 = 2.7937 \times 10^3$, $a_1 = 463.6798$, $\beta_0 = 1.5$, $\beta_1 = 1.7020$, $\gamma_0 = 1.9430$, $\gamma_1 = 1.5693$, $\theta_0 = \pi/4$. It is worth noting that such a field is anisotropic and not exactly self-similar (for a self-similar field β_0 would be equal to β_1 [4]). The structure function corresponding to this model is represented in Fig. 3. We remark a very good fit between the model and the data. The resulting interpolated map (1084×1084 points) was computed according to Equation (2) and is represented in Fig. 4. Note that these interpolation results may also be useful to provide low-complexity methods for synthesizing fields with stationary increments.

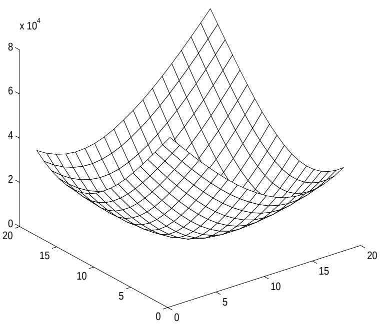

Figure 3: Modelled structure function.

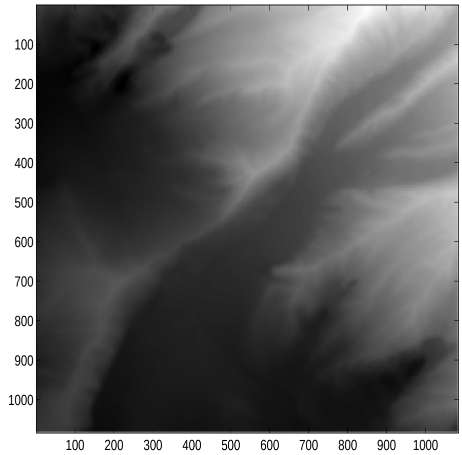

Figure 4: Interpolated map.

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