LINEAR CONSTRAINED REDUCED RANK AND POLYNOMIAL ORDER METHODS

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ABSTRACT

The Subspace-based Reduced Rank and Polynomial Order (RRPO) methods were proposed recently [1, 2, 3], which estimate a reduced order linear prediction polynomial whose roots are the desired "signal roots". In this paper, we describe how to extend the RRPO methods to include constraints involving known signal information. Simulation results indicate that by incorporating known signal information such as source direction angle, the estimation of unknown source directions can be significantly improved, especially when the unknown source is weak, closely spaced and highly coherent with the known source.

1. INTRODUCTION

Constrained MUSIC was presented in [4] as a method of incorporating information regarding known source directions. Knowledge of a source direction is equivalent to knowledge of one of the dimensions of the signal subspace. By constraining the signal subspace to include this dimension, the performance of source location estimates can be significantly enhanced.

The subspace-based Reduced Rank and Polynomial Order (RRPO) methods were proposed recently [1, 2, 3], which estimates an r^{th} order linear prediction polynomial (r: the number of signals) whose roots are the desired "signal roots". Simulation results [1, 3] indicate that the performance of the RRPO methods approach that of Root-MUSIC or Minimum-Norm, but with computational complexity of $O(mr^3)$ (m: the number of array elements) as opposed to $O(m^2)$ for the subspace-based method with full polynomial order. Savings are achieved provided $r^3 < m$. RRPO is similar to MODE method previously proposed by Stoica *et al* [5]. However variations on RRPO, such as weighting and model overfitting [3], yield similar performance to MODE without the costly second step of MODE.

In this paper, we describe how to extend the Reduced Rank and Polynomial Order (RRPO) methods to include constraints involving known signal information. The usefulness of the proposed constrained RRPO methods is demonstrated by an application to DOA findings over a wide range of scenarios. Simulation results indicate that by incorporating known signal information such as source direction angle, the estimation of unknown source directions can be significantly improved, especially when the unknown source is weak, closely spaced and highly coherent with the known source. For a more detailed discussion of constrained MUSIC approach, we refer to [4, 6, 7].

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2. BACKGROUND

The array output snapshot vector can be modeled as

$$\mathbf{x}(k) = \mathbf{A}\mathbf{s}(k) + \mathbf{n}(k). \tag{1}$$

With this data model, the correlation matrix $\mathbf{R}$ takes the form

$$\mathbf{R} = E\left[\mathbf{x}(k)\mathbf{x}(k)^H\right] = \mathbf{A}\mathbf{P}\mathbf{A}^H + \sigma^2\mathbf{I}. \tag{2}$$

The EVD of the matrix $\mathbf{R}$ can be partitioned as

$$\mathbf{R} = [\mathbf{U}_s \;\; \mathbf{U}_n] \begin{bmatrix} \Lambda_s & \mathbf{0} \\ \mathbf{0} & \sigma^2\mathbf{I}_{m-r} \end{bmatrix} [\mathbf{U}_s \;\; \mathbf{U}_n]^H. \tag{3}$$

Let $b(z) = b_0 + b_1 z^{-1} + b_2 z^{-2} + \ldots + b_r z^{-r}$ be an r^{th} degree polynomial with r roots at $e^{j\omega_1}, e^{j\omega_1}, \ldots, e^{j\omega_r}$, respectively, $\mathbf{b} = [\; b_0 \;\; b_1 \;\; \ldots \;\; b_r \;]^T \in \mathbf{C}^{r+1}$. Next let $\mathbf{B}$ be the $m \times (m-r)$ Toeplitz matrix defined by

$$\mathbf{B}^H = \begin{bmatrix} b_r & b_{r-1} & \ldots & b_0 & & \mathbf{0} \\ & \ddots & \ddots & & \ddots & \\ \mathbf{0} & & b_r & b_{r-1} & \ldots & b_0 \end{bmatrix}. \tag{4}$$

Then the following important property is observed

$$\mathbf{A}^H\mathbf{B} = \mathbf{0} \tag{5}$$

where $\mathbf{A} = [\; \mathbf{a}(\omega_1) \;\; \mathbf{a}(\omega_2) \;\; \ldots \;\; \mathbf{a}(\omega_r) \;] \in \mathbf{C}^{m\times r}$ is an unknown angular frequency matrix with $\mathbf{a}(\omega) = [\; 1 \;\; e^{j\omega} \;\; \ldots \;\; e^{j\omega(m-1)} \;]^T \in \mathbf{C}^m$. We also know that

$$range(\mathbf{U}_s) = range(\mathbf{A}). \tag{6}$$

This results in the following key equation:

$$\mathbf{U}_s^H\mathbf{B} = \mathbf{0} \Longleftrightarrow \mathbf{u}_i^H\mathbf{B} = \mathbf{0} \qquad i = 1, \ldots, r; \tag{7}$$

3. LINEAR CONSTRAINED RRPO METHODS

In this section, we will describe how to extend the Reduced Rank and Polynomial Order (RRPO) methods to include constraints involving known signal information.

3.1. Improved Subspace Estimation By Using Constraints

Assume that the directions of p ($p < r$) signals are known, with directions θ_1, ..., θ_p. Since array is assumed to be calibrated (i.e., $\mathbf{a}(\theta)$ is a known function of direction angle θ), the assumption that directions of p signals are known is equivalent to the knowledge of the constraint matrix $\mathbf{A}_c$ whose columns are the steering vectors corresponding to the signals with known directions of arrival:

$$\mathbf{A}_c = [\ \mathbf{a}(\theta_1) \quad \mathbf{a}(\theta_2) \quad \ldots \quad \mathbf{a}(\theta_p)\ ] \in \mathbf{C}^{m\times p}. \tag{8}$$

Taking the QR decomposition of $\mathbf{A}_c$, we have

$$\mathbf{A}_c = \begin{bmatrix} \mathbf{Q}_1 & \mathbf{Q}_2 \end{bmatrix} \begin{bmatrix} \mathbf{R}_1 \\ \mathbf{0} \end{bmatrix} \tag{9}$$

where $\mathbf{R}_1$ is a $p\times p$ upper triangular matrix, $\mathbf{Q}_1$ is the $m\times p$ matrix containing an orthonormal basis for the column span of the constraint matrix $\mathbf{A}_c$

$$span(\mathbf{A}_c) = span(\mathbf{Q}_1) \tag{10}$$

and $\mathbf{Q}_2$ is the $m\times(m-p)$ matrix whose column span is in the null space of $\mathbf{A}_c^H$

$$\mathbf{Q}_2^H\mathbf{A}_c = \mathbf{0}. \tag{11}$$

Then the vector $\mathbf{y}(k)$, which is the result of projecting snapshot vector $\mathbf{x}(k)$ into the null space of $\mathbf{A}_c^H$, may be expressed as

$$\mathbf{y}(k) = \mathbf{P}_{\mathbf{A}_c^\perp}\mathbf{x}(k) = \mathbf{Q}_2\mathbf{Q}_2^H\mathbf{x}(k). \tag{12}$$

Before we proceed futher, let us partition the matrix $\mathbf{A}$, and the vector $\mathbf{s}(k)$ as follows:

$$\mathbf{A} = [\ \mathbf{A}_c \quad \mathbf{A}_{uc}\ ] \in \mathbf{C}^{m\times r} \tag{13}$$

and

$$\mathbf{s}(k) = [\ \mathbf{s}_c^T(k) \quad \mathbf{s}_{uc}^T(k)\ ]^T \in \mathbf{C}^r \tag{14}$$

where the $m\times(r-p)$ matrix $\mathbf{A}_{uc}$ is the matrix whose columns are the steering vectors corresponding to the signals with unknown directions of arrival. Similarly, vector $\mathbf{s}_c(k)$ of length p, and vector $\mathbf{s}_{uc}(k)$ of length $(r-p)$ represent signal modulation vectors corresponding to known and unknown signals, respectively. With the above definition, we can continue our derivation of $\mathbf{y}(k)$ as follows:

$$\mathbf{y}(k) = \mathbf{Q}_2\mathbf{Q}_2^H\mathbf{x}(k) = \mathbf{Q}_2\mathbf{Q}_2^H(\mathbf{A}_{uc}\mathbf{s}_{uc}(k) + \mathbf{n}(k)) \tag{15}$$

where we use (11) to obtain the last equality. From the last equation, it is clearly shown that the effect of projecting the snapshot vector $\mathbf{x}(k)$ into the null space of $\mathbf{A}_c^H$ is that the signals with known directions of arrival information have been removed. Define $\mathbf{Y}$ as the data matrix containing N transformed snapshot vectors

$$\mathbf{Y} = [\ \mathbf{y}(1) \quad \mathbf{y}(2) \quad \ldots \quad \mathbf{y}(N)\ ] = \mathbf{Q}_2\mathbf{Q}_2^H\mathbf{X}. \tag{16}$$

The subspace information relating to the matrix $\mathbf{Y}$ can be obtained either from SVD of $\mathbf{Y}$ or from the EVD of the estimated correlation matrix $\hat{\mathbf{R}}_y$ defined as:

$$\hat{\mathbf{R}}_y = \frac{1}{N}\sum_{k=1}^{N}\mathbf{y}(k)\mathbf{y}(k)^H = \mathbf{Q}_2\mathbf{Q}_2^H\hat{\mathbf{R}}_x\mathbf{Q}_2\mathbf{Q}_2^H \tag{17}$$

with $\hat{\mathbf{R}}_x = \frac{1}{N}\sum_{k=1}^{N}\mathbf{x}(k)\mathbf{x}(k)^H$. Since the matrix $\mathbf{Y}$ is obtained by projecting the matrix $\mathbf{X}$ into the orthogonal complement of the p dimensional space spanned by the constraint vectors, the rank of the matrix $\hat{\mathbf{R}}_y$ is clearly equal to $(m-p)$. It follows that matrix $\hat{\mathbf{R}}_y$ has p zero eigenvalues. The remaining $(m-p)$ eigenvalues are arranged in decreasing order. Let $\hat{\mathbf{\Lambda}}_{cs2}$ denote the diagonal matrix containing the largest $(r-p)$ nonzero eigenvalues of $\hat{\mathbf{R}}_y$, and $\hat{\mathbf{\Lambda}}_{cn}$ denote the diagonal matrix containing the smallest $(m-r)$ nonzero eigenvalues. Using this notation, the EVD of $\hat{\mathbf{R}}_y$ can be written as

$$\hat{\mathbf{R}}_y = \begin{bmatrix} \hat{\mathbf{U}}_{cs1} & \hat{\mathbf{U}}_{cs2} & \hat{\mathbf{U}}_{cn} \end{bmatrix} \begin{bmatrix} \mathbf{0} & & \\ & \hat{\mathbf{\Lambda}}_{cs2} & \\ & & \hat{\mathbf{\Lambda}}_{cn} \end{bmatrix} \begin{bmatrix} \hat{\mathbf{U}}_{cs1}^H \\ \hat{\mathbf{U}}_{cs2}^H \\ \hat{\mathbf{U}}_{cn}^H \end{bmatrix} \tag{18}$$

where $m\times p$ matrix $\hat{\mathbf{U}}_{cs1}$ represents p dimensional constrained (known) portion of signal subspace, $m\times(r-p)$ matrix $\hat{\mathbf{U}}_{cs2}$ represents $(r-p)$ dimensional unconstrained (unknown) portion of signal subspace, and $m\times(m-r)$ matrix $\hat{\mathbf{U}}_{cn}$ represents $(m-r)$ dimensional noise subspace. So the signal subspace and noise subspace relating to the matrix $\mathbf{Y}$ can now be found in $\hat{\mathbf{U}}_{cs} = [\ \hat{\mathbf{U}}_{cs1} \quad \hat{\mathbf{U}}_{cs2}\ ] \in \mathbf{C}^{m\times r}$ and $\hat{\mathbf{U}}_{cn} \in \mathbf{C}^{m\times(m-r)}$, respectively.

3.2. Linear Constrained RRPO Methods

After obtaining the subspace estimates by using constraints, we can use them in the RRPO methods. In this section, we will present summary on how to use them to produce the various linear constrained RRPO methods. First define operator $\mathcal{T}$ as

$$\mathcal{T}(\mathbf{u}_i, t) = \begin{bmatrix} u_{t+1}^{(i)} & u_{t+2}^{(i)} & \ldots & u_m^{(i)} \\ u_t^{(i)} & u_{t+1}^{(i)} & \ldots & u_{m-1}^{(i)} \\ \vdots & \vdots & \ldots & \vdots \\ u_1^{(i)} & u_2^{(i)} & \ldots & u_{m-t}^{(i)} \end{bmatrix} \tag{19}$$

where $\mathbf{u}_i$, the i^{th} eigenvector from the signal subspace $\hat{\mathbf{U}}_{cs}$, is defined as $\mathbf{u}_i = [\ u_1^{(i)} \quad u_2^{(i)} \quad \ldots \quad u_m^{(i)}\ ]^T \in \mathbf{C}^m$.

Linear Constrained RRPO Method

1. Construct matrix $\mathbf{U}_i = \mathcal{T}(\mathbf{u}_i, r)$ for $i = 1$ to r using the i^{th} eigenvector $\mathbf{u}_i$.
2. The polynomial coefficient vector $\mathbf{b}$ is calculated using:

   a. EVD based method: Construct matrix $\mathbf{Q}$ based on

   $$Q = \sum_{i=1}^{r}\mathbf{U}_i\mathbf{U}_i^H \tag{20}$$

   and $\mathbf{b}$ is the eigenvector associated with the smallest eigenvalue of the matrix Q.

   b. SVD based method: Construct matrix $\mathbf{\Gamma}$ based on

   $$\mathbf{\Gamma} = \begin{bmatrix} \mathbf{U}_1^* & \mathbf{U}_2^* & \ldots & \mathbf{U}_r^* \end{bmatrix}^T \tag{21}$$

   where $*$ denotes complex conjugate. Then $\mathbf{b}$ is the right singular vector of $\mathbf{\Gamma}$ corresponding to the minimum singular value.

3. Frequency or DOA estimates can be deduced from the roots of the polynomial $b(z)$ associated with vector $\mathbf{b}$.

Linear Constrained RRPO with Model Overfitting

1. Construct matrix $\mathbf{U}_i = \mathcal{T}(\mathbf{u}_i, q)$ for $i = 1$ to r using the i^{th} eigenvector $\mathbf{u}_i$. and the matrix $\mathbf{G}$ based on

$$\mathbf{G} = \left[\begin{array}{cccc} \mathbf{U}_1^* & \mathbf{U}_2^* & \ldots & \mathbf{U}_r^* \end{array} \right]^T . \tag{22}$$

2. The overfitting coefficient vector $\mathbf{c}$, which is defined as $\mathbf{c} = [\begin{array}{cccc} c_0 & c_1 & \ldots & c_q \end{array}]^T \in \mathbf{C}^{q+1}$ with overfitting order $q > r$, can be obtained by solving $\mathbf{Gc} = \mathbf{0}$ using LS or TLS method.

3. Frequency or DOA estimates can be deduced from the signal roots of the polynomial $c(z)$ associated with vector $\mathbf{c}$.

Linear Constrained RRPO with Noise Subspace Transformation

1. Convert the matrix $\hat{\mathbf{U}}_{cn}$ into banded matrix $\hat{\mathbf{B}}$

$$\begin{aligned} \hat{\mathbf{U}}_{cn}\mathbf{Q} &= \left[\begin{array}{ccc} b_{1,r} & & \mathbf{0} \\ b_{1,r-1} & \ddots & \\ \vdots & \ddots & b_{m-r,r} \\ b_{1,0} & & b_{m-r,r-1} \\ & \ddots & \vdots \\ \mathbf{0} & & b_{m-r,0} \end{array} \right] \\ &= \left[\begin{array}{cccc} \mathbf{Jb}_1 & & & \mathbf{0} \\ & \mathbf{Jb}_2 & & \\ & & \ddots & \\ \mathbf{0} & & & \mathbf{Jb}_{m-r} \end{array} \right] \\ &\triangleq \hat{\mathbf{B}} \end{aligned} \tag{23}$$

where $\mathbf{J}$ is permutation matrix, and get an estimate of $\mathbf{b}$ by averaging.

$$\hat{\mathbf{b}} = \frac{1}{m-r} \sum_{i=1}^{m-r} \mathbf{b}_i \tag{24}$$

2. Frequency or DOA estimates can be deduced from the roots of the polynomial $b(z)$ associated with vector $\mathbf{b}$.

4. SIMULATION RESULTS

For the simulations, a two source data model is used. We assume that the source direction angle θ_1 is known and θ_2 is unknown. The random complex amplitudes for two signals, $\alpha_i(k), i = 1, 2$ are appropriately randomized to produce various types of coherence. The signal-to-noise ratio (SNR) of the unknown signal is defined as

$$SNR = \frac{E(|\alpha_2(k)|^2)}{\sigma^2} \tag{25}$$

where σ^2 is the white noise power (variance), and $E[.]$ is the expectation operator. For all simulations, we use the following parameters and assumptions, unless otherwise specified: $\theta_1 = 90^o$, $\theta_2 = 92^o$, and $\alpha_1(k) = \alpha_2(k)$. We assume that the array is linear and the sensors are equispaced with half wavelength spacing. The number of sensors is $m = 10$ and the number of snapshot vectors is $n = 100$ for each Monte Carlo trial. One thousand trials are used to calculate the root mean squared error (RMSE) of θ_2. The following algorithms are implemented in our simulations for comparisons: RRPO with Weighting (RRPOW), Total Least Square (TLS), MUSIC (MUSIC), Linear Constrained RRPO (LCRRPO), Linear Constrained RRPO with Overfitting (LCRRPOO), Linear Constrained Total Least Square (LCTLS), and Linear Constrained MUSIC (LCMUSIC).

4.1. RMSE versus SNR

In Figures 1, 2, 3, we plot the RMSE of θ_2 estimate versus SNR for various levels of coherence, $\rho = .999, .99, 0$, respectively. The following observations are obtained.
(a). The coherence between two sources has little effect on the estimation performance of constrained RRPO methods. For incoherent sources, the estimation performance of unconstrained RRRPO method approaches that of constrained RRPO method. This is also observed for the constrained TLS and MUSIC.
(b). When the SNR is beyond a certain threshold value (in our case, for example, $SNR > 0dB$), the performance of LCRRPOO is very close to that of LCTLS.

4.2. RMSE versus Source Separation

In Figures 4, 5, 6, we plot the RMSE of θ_2 estimate versus source separations for various levels of coherence, $\rho = .999, .99, 0$, respectively. $\theta_1 = 90^o$, and θ_2 is varied from $\theta_2 = 91^o$ to $\theta_2 = 115^o$. The following observations are obtained.
(a). As previously mentioned, the coherence between two sources has little effect on the estimation performance of constrained RRPO methods. For incoherent sources, the estimation performance of unconstrained RRRPO method approaches that of constrained RRPO method.
(b). When the SNR is beyond a certain threshold value (in this test, for example, SNR=30 dB), the performance of LCRRPOO is very close to that of LCTLS when source separation is small (in this test, for example, $\Delta\theta < 17^o$), and better than that of LCTLS and approaches that of LCMUSIC when source separation is large (in this test, for example, $\Delta\theta > 17^o$).
(c). If sources are highly correlated or coherent (for example $\rho = .999$ or $.99$), when source separation is larger than a certain threshold (13^o in our example), the performance of RRPOW method is equivalent to that of LCRRPO, and approaches that of LCTLS.

5. CONCLUSION

In this paper, the RRPO methods are extended to include constraints involving known signal information. Simulation results indicate that by incorporating known signal information such as source direction angle, the estimation of unknown source directions can be significantly improved, especially when the unknown source is weak, closely spaced and highly coherent with the known source.

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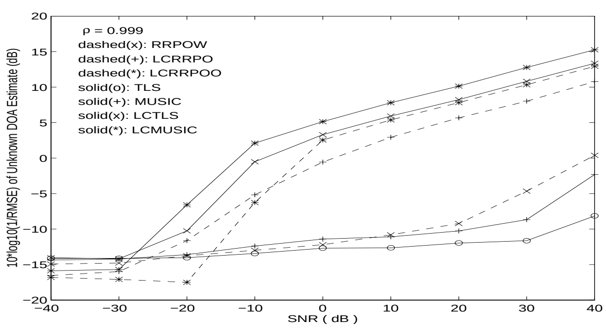

Figure 1: RMSE versus SNR. $\rho = .999$.

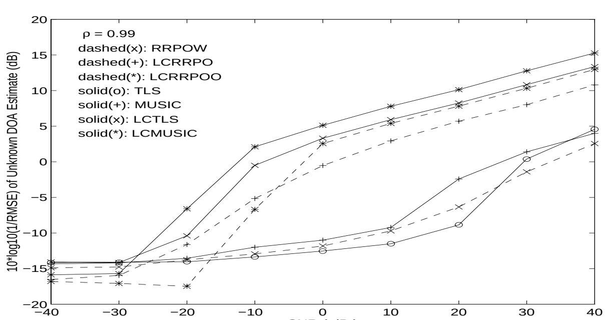

Figure 2: RMSE versus SNR. $\rho = .99$.

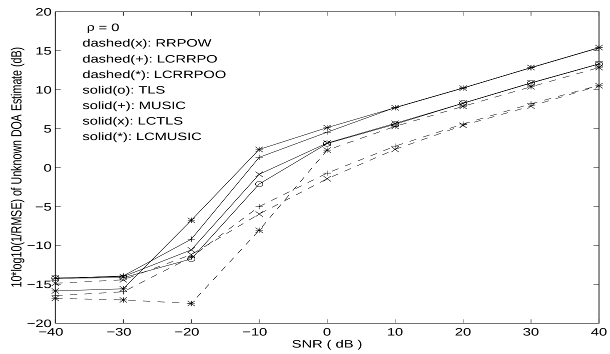

Figure 3: RMSE versus SNR. $\rho = 0$.

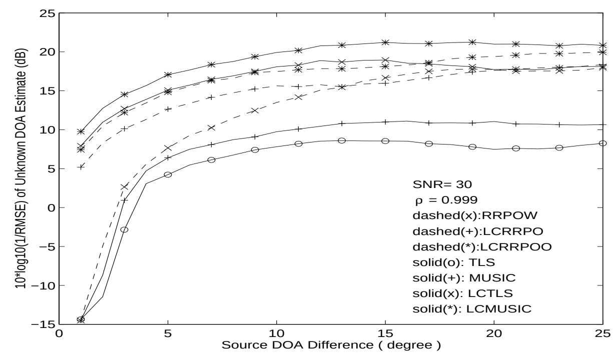

Figure 4: RMSE versus source separation. $\rho = .999$.

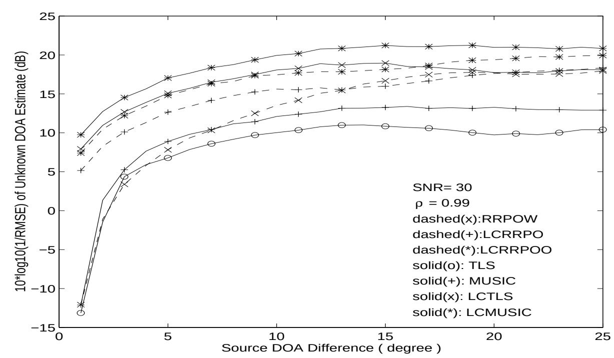

Figure 5: RMSE versus source separation. $\rho = .99$.

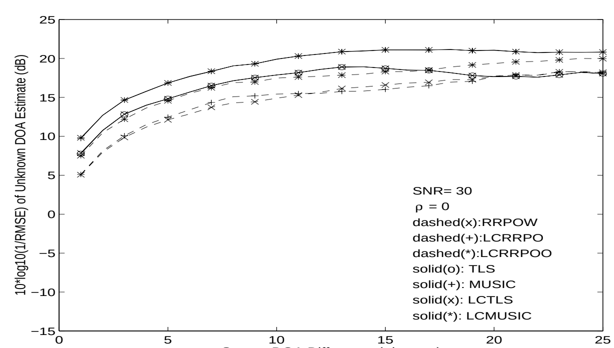

Figure 6: RMSE versus source separation. $\rho = 0$.