ROBUST ADAPTIVE SUBSPACE DETECTORS FOR SPACE TIME PROCESSING

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ABSTRACT

In this paper we consider the problem of detecting a subspace signal when there is uncertainty in the subspace. Such uncertainty usually causes a mismatch between the detector and the signal to be detected, which may lead to significant loss in performance. To improve the robustness of the detection procedure we apply robust adaptive subspace detectors based on extending the dimension of the signal subspace. We consider two types of adaptive constant false alarm rate (CFAR) detector structures for the extended subspace detectors: CFAR generalized likelihood ratio detector (CFAR GLR) and CFAR matched subspace detector (CFAR MSD). Using Monte-Carlo simulations, we study the performance of the robust adaptive subspace detectors for space-time processing.

1. INTRODUCTION

The problem of detecting subspace signals is of interest in radar and sonar signal processing. The term *subspace signal* describes an L dimensional vector which is known to belong to a given subspace of dimension $p < L$. When the waveform of the signal is completely known $p = 1$. For the case where the signal subspace is not completely known, several authors have suggested to improve the robustness of the detector by increasing the dimension of the signal subspace [1], [2]. In this paper we consider robust adaptive detectors based on extended signal subspace and analyze their performance.

We consider the following detection problem: We are given L complex data samples, $\mathbf{y} = [y(0), \cdots, y(L-1)]^T$. These data samples may represent L samples of a scalar time series, or (as in the space time processing application) N samples of an array of M elements with $L = MN$. Based on these data, the detector must decide between two possible hypotheses: According to the null hypothesis H_0 the data consists of noise $\mathbf{v}$ only, while according to the alternative hypothesis H_1 the data consists of the sum of the signal $\mu\mathbf{x}$ and noise $\mathbf{v}$, $\mathbf{y} = \mu\mathbf{x} + \mathbf{v}$. The signal $\mathbf{x}$ obeys the linear subspace model, $\mathbf{x} = H\boldsymbol{\theta}$ where $H \in C^{L\times p}$ and $\boldsymbol{\theta} \in C^p$. The noise is complex Gaussian with zero mean and covariance $\sigma^2 R$.

Most studies of the above problem have assumed that the signal subspace matrix H is completely known. Recently, the case of unknown signal subspace matrix was studied in [1]. However, the analysis there is limited to the case of known noise covariance R. In this paper we extend the analysis to the more realistic case of unknown noise covariance.

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2. ADAPTIVE SUBSPACE DETECTORS

In this section we review and present some results on adaptive subspace detectors.

When both R and H are known the generalized likelihood ratio (GLR) test of H_0 versus H_1, for unknown noise variance σ^2, is [3]

$$\phi(\mathbf{y}) = \begin{cases} 1 & t_0(\mathbf{y}) > \phi_0 \\ 0 & t_0(\mathbf{y}) < \phi_0 \end{cases} \tag{1}$$

where 1 stands for H_1, 0 stands for H_0, and

$$t_0(\mathbf{y}) = \frac{\mathbf{y}^* R^{-1} H (H^* R^{-1} H)^{-1} H^* R^{-1} \mathbf{y}}{\mathbf{y}^* R^{-1} \mathbf{y}} \tag{2}$$

This GLR detector is uniformly most powerful (UMP) invariant within a certain class of tests and hence optimal [4]. In addition it has constant false alarm rate (CFAR) independent of the noise variance σ^2.

When R is unknown, an adaptive subspace detector may be obtained by replacing R by its estimate $\hat{R}$. This type of detector was suggested in [3]. Following [3] we will refer to this detector as constant false alarm rate

(CFAR) matched subspace detector (MSD) or simply CFAR MSD.

The adaptive CFAR MSD is not equivalent to the generalized likelihood ratio test for this problem. The GLR test for one-dimensional signal subspace was derived by Kelly in [5]. [2] extends Kelly's derivation for arbitrary dimension $p < L$ of the signal subspace. In the following we derive a new expression for the GLR test statistic, which is considerably simpler than the one in [2]. As in [5] and [2] we assume that noise covariance matrix is estimated from a set of K secondary complex L-vectors, $\mathbf{y}_k$, $k = 1, \cdots, K$. The secondary vectors are assumed to be mutually independent and to have the same statistical properties as the interference in the primary vector $\mathbf{y}$.

It follows from the derivation in [5] and [2] that the GLR test statistic for known signal (known $\boldsymbol{\theta}$) is given by,

$$\ell(\mathbf{y}|\boldsymbol{\theta}) = \frac{1 + \mathbf{y}^* S^{-1}\mathbf{y}}{1 + (\mathbf{y} - H\boldsymbol{\theta})^* S^{-1}(\mathbf{y} - H\boldsymbol{\theta})} \tag{3}$$

where $S = \sum_{k=1}^{K} \mathbf{y}_k \mathbf{y}_k^*$.

To obtain the GLR test statistic we need an estimate of $\boldsymbol{\theta}$ which maximizes (3). The maximum is attained when the quadratic $(\mathbf{y} - H\boldsymbol{\theta})^* S^{-1}(\mathbf{y} - H\boldsymbol{\theta})$ is minimized with respect to $\boldsymbol{\theta}$. The quadratic has the form of a weighted least squares cost function and the minimizing $\boldsymbol{\theta}$ is the weighted least squares solution,

$$\boldsymbol{\theta}_0 = (H^* S^{-1} H)^{-1} H^* S^{-1}\mathbf{y} \tag{4}$$

Substituting (4) into (3) we get the GLR test statistic

$$\ell(\mathbf{y}) = \frac{1 + \mathbf{y}^* S^{-1}\mathbf{y}}{1 + \mathbf{y}^H S^{-1}\mathbf{y} - \mathbf{y}^* S^{-1} H (H^* S^{-1} H)^{-1} H^* S^{-1}\mathbf{y}} \tag{5}$$

A somewhat simpler equivalent test may be obtained by defining the quantity $\eta(\mathbf{y})$

$$\eta(\mathbf{y}) = \frac{\mathbf{y}^* S^{-1} H (H^* S^{-1} H)^{-1} H^* S^{-1}\mathbf{y}}{1 + \mathbf{y}^H S^{-1}\mathbf{y}} \tag{6}$$

As $\ell(\mathbf{y}) = 1/(1 - \eta(\mathbf{y}))$ the test $\ell(\mathbf{y}) > \ell_0$ is equivalent to the test $\eta(\mathbf{y}) > \eta_0$ where $\eta_0 = (\ell_0 - 1)/\ell_0$.

Both the adaptive CFAR MSD and the adaptive CFAR GLR subspace detectors are based on the assumption that the signal subspace is known. In the following section we consider the case where there is uncertainty in the signal subspace, and introduce robust versions of these detectors.

3. EXTENDED SUBSPACE DETECTORS

When the signal matrix H is not known, the detector uses instead an assumed matrix $F \in C^{L \times N_s}$. The performance of the mismatched detector deteriorates rapidly as the mismatch between the assumed and actual signal subspaces grows, as illustrated in the example of Figure 1. For the case of known noise covariance we have shown in [1] that robust detectors may be obtained by increasing the dimension of the assumed signal subspace and properly designing the matrix F. In the following we discuss the design of robust detectors for the case of unknown noise covariance. To enhance the clarity of the paper, we present the discussion in the context of the space time processing application. The results may readily be applied to a wide range of subspace detection problem.

3.1. Application to Space Time Adaptive Processing

In space time processing the problem of detecting a target at a given bearing and with a given radial velocity may be formulated as a subspace signal detection problem where the signal subspace has dimension $p = 1$ and the signal matrix H becomes [6],

$$H = \mathbf{h} = \mathbf{b}(\omega) \otimes \mathbf{a}(\alpha, \theta) \tag{7}$$

$\otimes$ denotes the Kronecker product operator. $\mathbf{a}$ is the $M \times 1$ spatial steering vector. Assuming that the radar antenna is a uniformly spaced linear array, $\mathbf{a}$ may be written as

$$\mathbf{a}(\psi) = [1, e^{j2\pi\psi}, \cdots, e^{j(M-1)2\pi\psi}]^T \tag{8}$$

$\psi = (d/\lambda_0)\cos\theta\sin\alpha$ is the normalized spatial frequency, θ is the elevation angle, α is the azimuth, d is the inter-element spacing and λ_0 is the wavelength corresponding to the center frequency of the radar.

$\mathbf{b}(\omega)$ is the $N \times 1$ temporal steering vector. ω is the normalized Doppler frequency of the target $\omega = f_d T_r$, where f_d is the Doppler frequency and T_r is the pulse repetition interval of the radar.

$$\mathbf{b}(\omega) = [1, e^{j2\pi\omega}, \cdots, e^{j(N-1)2\pi\omega}]^T \tag{9}$$

In practice it is not possible to test for the existence of a target in all bearings and Doppler frequencies. The set of all possible bearings and Doppler frequencies is represented by a finite grid. For each cell in this grid, the task of the detector is to test for the existence of the target in this cell. In this scenario, however, the target signal can no longer be represented by a one-dimensional subspace as in Eq. (7). Then, it is intuitively appealing to replace the one-dimensional subspace by a subspace of a larger dimension $N_s > 1$. This leads to the following signal model

$$\mathbf{x} = F\boldsymbol{\beta} \tag{10}$$

where $F \in C^{L \times N_s}$ and $\boldsymbol{\beta} \in C^{N_s}$.

The resulting test statistic for the adaptive CFAR MSD is given by

$$T(\mathbf{y}) = \frac{\mathbf{y}^* S^{-1} F (F^* S^{-1} F)^{-1} F^* S^{-1} \mathbf{y}}{\mathbf{y}^* S^{-1} \mathbf{y}} \tag{11}$$

and the test statistic for the adaptive CFAR GLR detector is

$$\mu(\mathbf{y}) = \frac{\mathbf{y}^* S^{-1} F (F^* S^{-1} F)^{-1} F^* S^{-1} \mathbf{y}}{1 + \mathbf{y}^* S^{-1} \mathbf{y}} \tag{12}$$

The performance of these detectors depends on the dimension of the extended signal subspace and the actual choice of the extended signal matrix F. In [1] we presented two different approaches for constructing the signal matrix F: the lattice-based approach and the eigen-analysis based approach. The eigen-analysis approach was first introduced in [2] for the problem of detecting a signal whose Doppler frequency is not known precisely. [1] extends the approach for space time processing where there is uncertainty both in the Doppler and the bearing of the target. Numerical experiments indicate that the eigen-based approach leads to a better detection performance, both for the known and unknown noise covariance cases. We therefore limit our discussion to the eigen-analysis based approach.

Let us represent the uncertainty in the target Doppler by letting ω be contained in the interval $(\omega_0 - \frac{\Delta\omega}{2}, \omega_0 + \frac{\Delta\omega}{2})$, where ω_0 is known and $\Delta\omega$ denotes the uncertainty in the Doppler frequency of the signal. Similarly, we represent the uncertainty in the spatial frequency by letting ψ be contained in the interval $(\psi_0 - \frac{\Delta\psi}{2}, \psi_0 + \frac{\Delta\psi}{2})$, where ψ_0 is known and $\Delta\psi$ denotes the uncertainty in the spatial frequency of the signal.

Assuming that the normalized Doppler frequency and the normalized spatial frequency of the signal are (individually) uniformly distributed in the corresponding uncertainty intervals we can express the matrix $R_x = E[\mathbf{x}\mathbf{x}^*]$ as

$$R_x = R_b \otimes R_a \tag{13}$$

where the elements of R_b and R_a are

$$\begin{aligned} R_b(n_1, n_2) &= \mathrm{sinc}[(n_1 - n_2)\Delta\omega] \\ R_a(m_1, m_2) &= \mathrm{sinc}[(m_1 - m_2)\Delta\psi] \end{aligned} \tag{14}$$

The matrix F is constructed as follows: For given values of the parameters $\Delta\psi$ and $\Delta\omega$ we evaluate the eigen values and eigen vectors of the above matrix R_x and use the eigen vectors corresponding to the dominant eigen values as basis vectors for the signal subspace. The number of dominant eigenvalues may be defined as the minimum number of eigen values whose sum exceeded ρL, where $0 < \rho < 1$ is close to unity (e.g. $\rho = 0.99$). All eigenvalues included in the sum are defined as dominant eigen values. Note that the sum of all eigenvalues of R_x is $\mathrm{trace}(R_x) = L$. The resulting number of dominant eigen values will be called the effective dimension of signal subspace.

We will refer to the resulting detectors as the robust adaptive CFAR MSD and the robust adaptive CFAR GLR. In the following we analyze and compare the performance of these detectors.

3.1.1. Performance Analysis

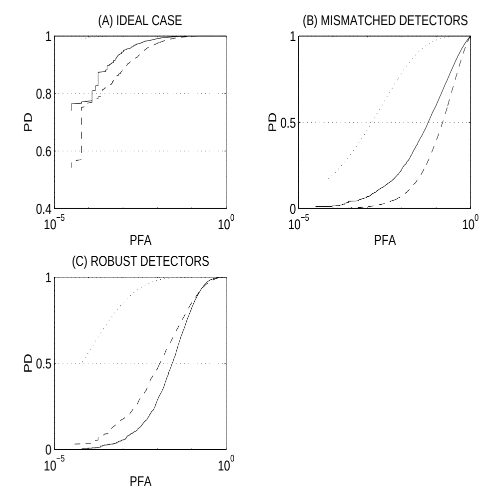

Figure 1: PD versus PFA for an accelerating target at uncertain direction. SNR = 15 dB, $\psi = \Delta\psi/2$. Adaptive CFAR GLR (solid line), Adaptive CFAR MSD (dashed line), CFAR GLR for known noise covariance (dotted line).

We performed Monte-Carlo simulations to study the performance of the robust adaptive subspace detectors. We assume that the array has four elements and the length of processing interval is four samples. The data has been generated assuming that $R = I$ where R is the noise covariance matrix. The number of snapshots for estimating the noise covariance is $K = 32$.

We considered two scenarios. In the first scenario the target has uncertain Doppler and bearing while the second scenario corresponds to a maneuvering target at an unknown bearing. For both scenarios the robust detectors were designed to detect a target of uncertain Doppler and bearing whose spatial frequency is in the interval $(-\Delta\psi/2, \Delta\psi/2)$ and whose temporal frequency is in $(-\Delta\omega/2, \Delta\omega/2)$ with $\Delta\omega = \Delta\psi = 0.3$. Due to space limitations, we will only present results for the scenario with the maneuvering target at an uncertain bearing. In this scenario the target has normalized spatial frequency $\psi = \Delta\psi/2$ and constant radial acceleration, so that its normalized temporal frequency increases linearly from $-\Delta\omega/2$ to $\Delta\omega/2$.

In Figure 1 we plot the probability of detection as a function of the false alarm probability for SNR of 15 dB. The SNR is defined as $\mathbf{x}^* R^{-1}\mathbf{x}$. Parts A and B of this figure demonstrates the need for robust detectors. Part A shows the receiver operation curves (ROC) for the case where the signal subspace is known at the detector. In this case both the adaptive CFAR GLR (solid line) and the adaptive CFAR MSD (dashed line) are perfectly matched to the signal. The ideally matched CFAR GLR for the case of known noise covariance (dotted line) is included for reference. As may be expected the adaptive CFAR GLR performs slightly better than the adaptive CFAR MSD. Part B of the figure shows the ROCs for the case of mismatch between the actual and assumed signal subspaces. In this case the subspace detectors are tuned to a target with $\psi = \omega = 0$. All three detectors exhibit significant performance losses relative to the corresponding matched detectors. As for the ideally matched detectors, the adaptive CFAR GLR performs better than the adaptive CFAR MSD.

In Part C we demonstrate the performance of the robust adaptive subspace detectors. The dimension of the extended subspace is $N_s = 4$. Contrary to the previous examples, it appears that in this case the robust adaptive CFAR MSD out performs the robust adaptive CFAR GLR. Comparing to Part B we observe that all robust detectors exhibit performance gains relative to their mismatched counterparts. Similar observations are obtained for the scenario in which the target has uncertain Doppler and bearing.

4. SUMMARY

We considered the problem of detecting a subspace signal when there is uncertainty in the signal subspace. Such uncertainty usually causes a mismatch between the detector and the signal to be detected, which may lead to significant loss in performance. To improve the robustness of the detection procedure we proposed robust subspace detectors based on extending the dimension of the signal subspace. We considered two types of adaptive constant false alarm rate (CFAR) detector structures for the extended subspace detectors: CFAR generalized likelihood ratio detector (CFAR GLR) and CFAR matched subspace detector (CFAR MSD). Using Monte-Carlo simulations, we studied the performance of the robust adaptive subspace detectors for the space-time processing application, focusing on the case of a maneuvering target at an uncertain bearing. In this application the target may be represented as a one-dimensional subspace signal. As may be expected, the one-dimensional ideally matched adaptive CFAR GLR slightly out performs its CFAR MSD counterpart. Similarly, the one-dimensional mismatched adaptive CFAR GLR exhibits better performance than its CFAR MSD counterpart. However, the results are different when the robust versions of these detectors are considered. It appears that the robust adaptive CFAR MSD out performs its CFAR GLR counterpart. Similar observations are obtained for the case where the target has uncertain Doppler and bearing

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