GLOBALLY OPTIMAL TWO CHANNEL FIR ORTHONORMAL FILTER BANKS ADAPTED TO THE INPUT SIGNAL STATISTICS†

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ABSTRACT

We introduce a new approach to adapt a 2-channel FIR orthonormal filter bank to the input second order statistics. The problem is equivalent to optimizing the magnitude squared response $F(e^{j\omega}) = |H(e^{j\omega})|^2$ of one the subband filters for maximum energy compaction under the constraint that $F(e^{j\omega})$ is Nyquist(2). The novel algorithm enjoys important advantages that are not present in previous work. First, we can ensure the positivity of $F(e^{j\omega})$ *over all frequencies* simultaneously with the Nyquist constraint. Second, for a fixed input power spectrum, the resulting filter $F_{opt}(z)$ is guaranteed to be a *global* optimum due to the convexity of the new formulation. The optimization problem is expressed as a multi-objective semi definite programming problem which can be solved efficiently and with great accuracy using recently developed interior point methods. Third, the new algorithm is extremely general in the sense that it works for any arbitrary filter order N and any given input power spectrum. Finally, obtaining $H_{opt}(e^{j\omega})$ from $F_{opt}(e^{j\omega})$ does not require an additional spectral factorization step.

1. INTRODUCTION

There has been a considerable interest in optimizing filter banks when quantizers are present [1, 2, 3, 4, 5, 6]. Given a fixed budget of b bits for the subband quantizers, the goal is to simultaneously optimize the analysis and synthesis filters and to choose a subband bit allocation strategy such that the average variance of the error $e(n)$ at the output of the filter bank is minimized.

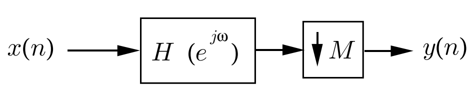

Figure 1: Schematic of the FIR energy compaction problem.

The energy compaction problem. Consider the scheme shown in Fig. 1. A wide-sense stationary (WSS) input $x(n)$ passes through an FIR filter $H(e^{j\omega})$ and is downsampled to produce an output $y(n)$. With the input power spectral density $S_{xx}(e^{j\omega})$ fixed, the compaction filter problem is to find $H(e^{j\omega})$ such that the variance of the output, given by

$$\sigma_y^2 = \int_{-\pi}^{\pi} |H(e^{j\omega})|^2 S_{xx}(e^{j\omega}) \frac{d\omega}{2\pi} \tag{1}$$

is maximized under the constraint

$$\frac{1}{M}\sum_{k=0}^{M-1} |H(e^{j(\omega-2\pi k/M)}))|^2 = |H(e^{j\omega})|^2|_{\downarrow M} = 1 \tag{2}$$

The constraint (2) means in particular that the magnitude squared response $|H(e^{j\omega})|^2$ is Nyquist(M). For the case of a two channel orthonormal filter bank ($M = 2$), the problems are equivalent: *optimizing one of the subband filters for maximum energy compaction is equivalent to optimizing a two channel orthonormal filter bank according to the input signal statistics.* Indeed, we can easily show that the coding gain of an orthonormal filter bank under optimum bit allocation and with the high bit rate quantizer assumptions is given by:

$$\mathcal{G}_{SBC}(2) = \frac{1}{\sqrt{G_{comp}(2,N)(2 - G_{comp}(2,N))}} \tag{3}$$

where $G_{comp}(2, N)$ is the so called **compaction gain** and is equal to $\sigma_{x_0}^2/\sigma_x^2$. Note that *the maximum possible compaction gain $G_{comp}(2, N)$ is equal to two whereas the coding gain $\mathcal{G}_{SBC}(2)$ can be arbitrarily large.*

2. THE PRODUCT FILTER APPROACH

From (1) and (2), we can immediately observe that the optimum solution, if it exists, is only a function of $|H(e^{j\omega})|^2$. By denoting **the product filter** $H(z)H(z^{-1})$ by $F(z)$, the output variance σ_y^2 in (1) can be rewritten as

$$\sigma_y^2 = r(0) + 2\sum_{n=1}^{N} f(n)r(n) \tag{4}$$

and the constraint (2) becomes

$$f(Mn) = \delta(n) \tag{5}$$

$$F(e^{j\omega}) \geq 0 \quad \forall\ \omega \tag{6}$$

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where $r(i)$ denotes the i^{th} autocorrelation coefficient of the input $x(n)$. The objective function is now linear in the optimization variables $f(n)$, $n \geq 1$ at the expense of an additional constraint, namely equation (6) which we shall refer to as the positivity constraint. The major difficulty with the product filter approach is to simultaneously satisfy the positivity and Nyquist constraints. A standard way to solve such optimization problems is to consider a finite set of discrete frequencies $\{\omega_i, 0 \leq i \leq L\}$ over the continuous frequency axis and enforce the positivity constraint only at those frequencies (see for example [7, 8]). The main problem with the "discretization" approach is that, in general, the resulting $G_{opt}(e^{j\omega})$ could be negative between the discrete frequencies *no matter how large L is* which, in turn, creates an infeasible spectral factorization step.

We show next, using a well known result from linear system theory, that the positivity constraint can be satisfied over all ω at the expense of $N(N+1)/2$ additional optimization variables.

3. THE STATE SPACE SOLUTION

Since $F(z) = H(z)H(z^{-1})$, the product filter is a two sided symmetric sequence and we can therefore write $F(z)$ as $D(z) + D(z^{-1})$. The function $D(z)$ completely characterizes $F(z)$ and it is natural to wonder whether the positivity condition on $F(e^{j\omega})$ can be reformulated in terms of some other condition(s) on $D(e^{j\omega})$. The answer turns out to be yes and is established by the well known **discrete time positive real lemma** [9]. We first start with a definition.

Definition 1. *Discrete positive real functions.* A square transfer matrix (function) $D(z)$ whose elements are real rational functions analytic in $|z| > 1$ is discrete positive real if, and only if, it satisfies all the following conditions :

$$\text{poles of } D(z) \text{ on } |z| = 1 \text{ are simple} \quad (7)$$

$$D(e^{j\omega}) + D(e^{-j\omega}) \geq 0 \quad \forall\, \omega \text{ at which } D(e^{j\omega}) \text{ exists} \quad (8)$$

Furthermore, if $z_0 = e^{j\omega_0}$, ω_0 real, is a pole of $D(z)$ and if $\mathbf{K}$ is the residue matrix of $D(z)$ at $z = z_0$, the matrix $Q = e^{-j\omega_0}\mathbf{K}$ is hermitian non negative definite.

Assume now that $D(z)$ has the following state space realization :

$$\begin{aligned} x(n+1) &= A_d x(n) + B_d u(n) \\ y(n) &= C_d x(n) + D_d u(n) \end{aligned} \quad (9)$$

where A_d is $N \times N$, B_d is $N \times P$, C_d is $L \times N$, and D_d is $L \times P$. For our case, $P = L = 1$. Then, the following lemma can be established.

Fact 1. *The discrete time positive real lemma* [9]. Let $D(z)$ be a transfer matrix (function) with real rational elements that is analytic in $|z| > 1$ with only simple poles on $|z| = 1$. Let (A_d, B_d, C_d, D_d) be a minimal realization of $D(z)$. Then, $D(z)$ is discrete positive real if, and only if, there exist a real symmetric positive definite matrix P_d and real matrices W_d and L_d such that :

$$P_d - A_d^T P_d A_d = L_d^T L_d \quad (10)$$

$$C_d^T - A_d^T P_d B_d = L_d^T W_d^T \quad (11)$$

$$D_d + D_d^T - B_d^T P_d B_d = W_d^T W_d \quad (12)$$

The above equalities (10-12) can be rewritten as the following matrix inequality :

$$\begin{bmatrix} P_d - A_d^T P_d A_d & C_d^T - A_d^T P_d B_d \\ C_d - B_d^T P_d A_d & D_d + D_d^T - B_d^T P_d B_d \end{bmatrix} \succeq 0 \quad (13)$$

where the notation $\succeq$ indicates that the above matrix should be positive semi-definite. Equation (13) represents therefore an equivalent condition for the positivity constraint. Assume now that $D(z)$ is implemented in a direct form structure with the following state space representation:

$$\begin{aligned} A_d &= \begin{bmatrix} 0 & I \\ 0 & 0 \end{bmatrix}, \quad B_d = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix} \\ C_d &= [\ f(N)\ \ldots\ f(1)\ ], \quad D_d = \frac{1}{2} \end{aligned} \quad (14)$$

Clearly, this state space realization is minimal since the number of delay elements is equal to the degree of $D(z)$. Then, the Nyquist constraint can be written as a linear equality constraint:

$$Q\ C_d^T = \mathbf{0} \quad (15)$$

where $\mathbf{0}$ is the zero vector and Q is a diagonal matrix with diagonal elements $\in \{0, 1\}$. The positions of the unity elements are determined by N. For example, for $N = 5$ and $M = 2$, the diagonal elements are $\{0\ 1\ 0\ 1\ 0\}$. In conclusion, we can represent the positivity constraint as a "linear" matrix inequality (LMI) whose entries are affine functions of the variables P_d and C_d, and the Nyquist constraint as an equality constraint on C_d. The optimization problem described by (4), (13) and (15) can be solved at this point. In specific, we can obtain a global optimum $C_{d_{opt}}$ and a feasible matrix P_d that will meet the constraints and maximize the objective function. We can then spectral factorize $F_{opt}(z)$ to obtain $H_{opt}(z)$ using any of the well known algorithms (see for example [10, pages 854–856]). It turns out however that the spectral factorization step can be completely avoided by writing the state space representation of the *minimum phase spectral factor*, $H_{min}(z)$, in terms of the matrices A_d, B_d, C_d, D_d and a particular P_d, namely the minimum element $P_{d_{min}}$ of the convex set of positive definite matrices satisfying equation (13) and (15). This is established in the next section.

4. THE SISO MINIMUM PHASE SPECTRAL FACTOR

Definition 2. *Minimum element.* We say that $P_{d_{min}} \in S$ is a minimum element of S with respect to the (strict) generalized inequality $\preceq$ $(\prec)$ if for every $P \in S$ we have $P_{d_{min}} \preceq (\prec) P$. Note that $P_{d_2} \succeq P_{d_1}$ is equivalent to $P_{d_2} - P_{d_1}$ is positive semi definite. If a set has a minimum element, this element is unique.

Theorem. Let $F(z) = D(z) + D(z^{-1})$ be a real rational function whose elements are analytic in $|z| > 1$. Assume that $D(z)$ satisfies the discrete time positive real lemma with a minimal realization $(A_d,\ B_d,\ C_d,\ D_d)$. In particular,

$F(e^{j\omega}) \geq 0 \ \forall\ \omega$. Then, the minimum phase spectral factor $H_{min}(z)$ of $F(z)$ can be expressed in the form:

$$H_{min}(z) = W_d + L_d^T(zI - A_d)^{-1}B_d \tag{16}$$

where

$$W_d = (D_d + D_d^T - B_d^T P_{min} B_d)^{1/2} \tag{17}$$

$$L_d^T = \frac{(C_d^T - A_d^T P_{min} B_d)}{(D_d + D_d^T - B_d^T P_{min} B_d)^{1/2}} \tag{18}$$

and $P_{d_{min}}$ is the minimum element in the convex set of symmetric positive definite matrices satisfying the LMI (13) and the Nyquist constraint (15). Alternatively, $P_{d_{min}}$ is also the unique solution to the following equations :

$$\begin{aligned} P_d &= A_d^T P_d A_d + (C_d^T - A_d^T P_d B_d) \\ &\quad (D_d + D_d^T - B_d^T P_d B_d)^{-1}(C_d^T - A_d^T P_d B_d)^T \end{aligned} \tag{19}$$

$$\begin{aligned} P_d &= A_1^T P_d A_1 + A_1^T P_d B_d \\ &\quad (R - B_d^T P_d B_d)^{-1} B_d^T P_d A_1 + C_d^T R^{-1} C_d \\ \text{where} \quad A_1 &= A_d - B_d R^{-1} C_d, \quad R = D_d + D_d^T \succ 0 \end{aligned} \tag{20}$$

The proof of all the above results can be found in [11]. Since A_d, B_d and D_d are already fixed by the choice (14), $H_{min}(z)$ is automatically obtained once the program returns C_d and $P_{d_{min}}$. We can include P_d in the objective function (4) but minimizing P_d directly will produce a vector valued objective function. Instead, we will minimize a scalar valued function of P_d with the help of the following observation.

Observation 1. Assume that $P_{d_{min}}$ is the minimum element in the convex set of symmetric positive definite matrices satisfying the LMI and Nyquist constraints (13) and (15). Then, $P_d = P_{d_{min}}$ if and only if $Tr(\mathbf{W}P_d)$ is minimum for every diagonal positive semi-definite matrix $\mathbf{W}$.

5. THE OPTIMIZATION ALGORITHM

The optimization problem reduces to the following final form:

$$\max_{C_d, P_d} \quad C_d\ \mathbf{R}^T - Tr(\mathbf{W}P_d) \tag{21}$$

where $\mathbf{R}^T = [r(N) \dots r(1)]^T$ and $\mathbf{W}$ is a diagonal positive semi definite weight matrix and find a real symmetric positive definite matrix such that

$$\begin{bmatrix} P_d - A_d^T P_d A_d & C_d^T - A_d^T P_d B_d \\ C_d - B_d^T P_d A_d & D_d + D_d^T - B_d^T P_d B_d \end{bmatrix} \succeq 0 \tag{22}$$

$$Q\ C_d^T = \mathbf{0} \tag{23}$$

and is therefore a maximization problem in the variable vector C_d and a minimization problem in the matrix P_d.

Observation 2. The optimization problem described by (21), (22) and (23) is a convex program in the variables C_d and P_d.

The above formulation is therefore a convex multi-objective optimization problem for which any local solution is also a global one. The particular choice of the trace function was intentional in order to use *semi definite programming*. Semi definite programs can be solved very efficiently both in theory and practice [12]. Two different programs are currently available at our web cite: the first one is written by Vandenberghe and Boyd [13] and uses a particular primal-dual interior point method. The second one uses the MATLAB LMI toolbox that implements the projective algorithm of Nesterov and Nemirovskii For more details, the reader is referred to *http://www.systems.caltech.edu/tuqan.*

6. ADDING REGULARITY CONSTRAINTS

The regularity property is important in wavelet applications and consists of forcing r zeros at $z = -1$. The first of these zeros ($r = 0$) is simply obtained from $F(-1) = 0$ (because $F(e^{j\omega}) \geq 0 \quad \forall\ \omega$, there will actually be a double zero at π). The second zero $r = 1$ is obtained by differentiating $F(e^{j\omega})$ twice with respect to ω, evaluating the result at π and setting it to zero. Repeating this procedure, we can easily derive the following set of equations:

$$D_d - C_d(A_d + I)^{-1}B_d = 0, \quad r = 0 \tag{24}$$

$$C_d \begin{bmatrix} (2N+1)^{2r} \\ \vdots \\ (2k+1)^{2r} \\ \vdots \\ 3^{2r} \end{bmatrix} = 0, \quad 1 \leq r < L \tag{25}$$

The two running programs assume the so called Slater conditions (existence of a strict feasible primal or dual) [12]. Unfortunately, adding the regularity constraint seem to violate those conditions indicating either an infeasible solution (which would be an interesting result) or more likely, a non strict primal and/or dual solution. We expect future versions of the programs to be able to relax the Slater conditions.

7. SIMULATION RESULTS

Example 1: AR(1) process. Assume that the input $x(n)$ is a zero mean AR(1) process with $R_{xx}(k) = \rho^{|k|}$ where $0 < \rho < 1$. The optimum compaction gain curves for $N = 2$ and 3 as a function of ρ are shown in Fig. 2. The curve for $N = 3$ coincide with the theoretical compaction gain formula $G_{comp}(2,3) = 1 + \dfrac{2\rho}{\sqrt{3+\rho^2}}$ derived in [6]. The precise difference is actually in the order of 10^{-10}. The last curve denotes the compaction gain when $N = \infty$ (ideal low pass filter case) and is equal to $G_{comp}(2,\infty) = \dfrac{4}{\pi} \arctan \dfrac{(1+\rho)}{(1-\rho)}$. From Fig. 2, it is therefore very clear that for an AR(1) process, we do not loose much by using short filters. Assume now that $\rho = 0.9$, $N = 3$, $M = 2$ and set $\mathbf{W} = \alpha \mathbf{I}, \alpha = 10^{-6}$. Although we do not have a formal proof that this particular choice of $\mathbf{W}$ will work for all inputs, we never had to change this setting over all the examples we have tried. The positivity of $F_{opt}(z)$ is demonstrated by the double roots shown in the Z-plane plot of Fig. 3(a). The spectral factorization is also quite accurate. The compaction gain remains almost the same and the positivity of $H_{min}(z)H_{min}(z^{-1})$ is preserved as we can see from Fig. 3(b).

Example 2: multiband AR(5) process. Assume that the input $x(n)$ is a zero mean multiband AR(5) process. The magnitude squared responses of the resulting optimum compaction filters are shown in Fig. 4 for $N = 7,\ 17$ and 27. Let $\mathbf{W} = \alpha\mathbf{I}$ where $\alpha = 10^{-6}$. The corresponding compaction gains are 1.524, 1.563 and 1.575. It can be verified that the positivity of the resulting filters is satisfied for all orders. Moreover, the accuracy of the spectral factorization is as good as in the previous example.

As a final remark, we emphasize that the formulation described in sections 3 and 5 works for any arbitrary value of M (not only $M = 2$) and can be used to generate globally optimal FIR energy compaction filters [11].

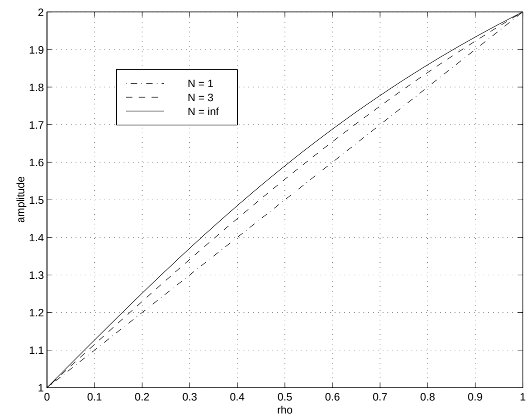

Figure 2: Compaction gain curves for an AR(1) process for $N = 2, 3$ and ∞ with $M = 2$.

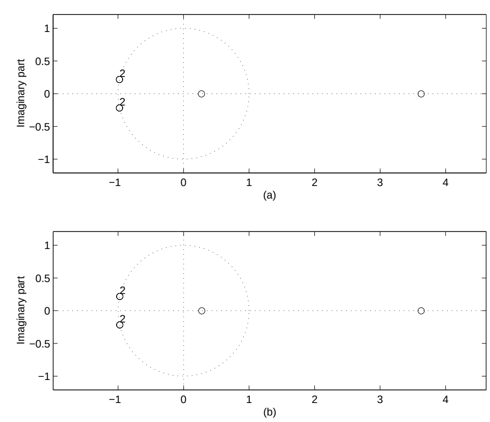

Figure 3: Double roots on the unit circle indicating the positivity of the product filter $F(z)$ (a) as the output of the program (b) as a result of convolving $h_{min}(n)$ with its flipped version.

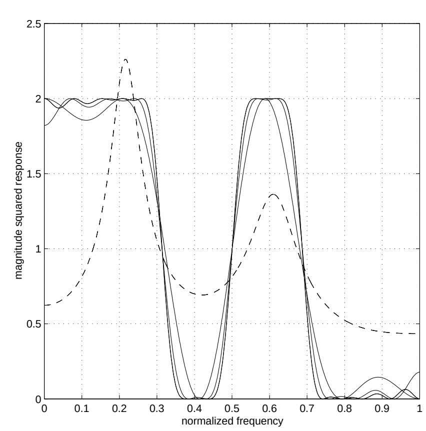

Figure 4: The magnitude squared responses of the optimum compaction filters corresponding to the multiband AR(5) process of order $N = 7,\ 17$ and 27 with $M = 2$.

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