CONTINUOUS-DILATION DISCRETE-TIME SELF-SIMILAR SIGNALS AND LINEAR SCALE-INVARIANT SYSTEMS

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ABSTRACT

In this paper we present a novel model for purely discrete-time self-similar processes and scale-invariant systems. The results developed are based on a new interpretation of the discrete-time scaling (equivalently dilation or contraction) operation which is defined through a mapping between discrete and continuous time. It is shown that it is possible to have continuous scaling factors through this operation even though the signal itself is discrete-time. We study both deterministic and stochastic discrete-time self-similar signals. We then derive the existence conditions of discrete-time deterministically self-similar signals with respect to some specific mappings. Finally, we discuss the construction of discrete-time linear scale-invariant (LSI) system and present results related to white noise driven system models of stochastic self-similar signals. Unlike continuous-time self-similar signals, it is possible to construct a wide class of non-trivial discrete-time self-similar signals.

1. INTRODUCTION

This paper addresses the problem of defining and representing discrete-time self-similar signals and systems.

The study of the discrete-time self-similar processes in this paper is motivated in part by the previous work of Wornell and colleagues [3, 4, 5] in continuous time. They provide formulations involving continuous-time, scale-invariant signals and systems. They also provide a detailed study of such systems for dyadic scale factors. Our paper here provides answers to questions such as: Is it possible to define purely discrete-time, self-similar signals? Are there formulations of discrete-time, scale-invariant systems? How do we provide a definition of dilation or scaling of discrete-time signal that is general enough to provide non-trivial self-similar signals and scale-invariant systems? The answer to the third question holds the key to answering the first two questions. A key result of this paper is the demonstration of the fact that it is possible to define scaling or dilation in such a way that is continuous even though the signal itself is discrete-time. Hereafter, we will use the term scaling exclusively to mean dilation. Using this definition of scaling, we develop definitions and constructions of deterministic and stochastic, discrete-time, self-similar signals and discrete-time scale-invariant systems.

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2. SCALING IN DISCRETE-TIME

2.1. Discrete-Time Scaling Operation

Generally the scaling or dilation operation of a discrete-time signal $x(n)$ by an arbitrary factor is not well defined. It is difficult to obtain an interpretation of scaling in the discrete-time domain that is as unambiguous as that in the continuous-time domain. Operations such as upsampling, interpolation, downsampling and fractional sampling rate alteration [2] can have a scaling interpretation. However, such operations cannot handle scaling factors over a continuum. We present here a different approach to discrete-time scaling that can handle continuous scaling factors. We define the discrete-time scaling operation in a way that effectively amounts to converting $x(n)$ into a continuous-time signal through an inversible mapping, applying the scaling operation to the continuous-time signal and finally inverse mapping the signal back to the discrete-time domain. The actual definition is based on operations in the frequency domain.

Let $f(t)$ be a continuous-time signal and $F(\Omega)$ its Fourier transform:

$$F(\Omega) = \mathcal{F}\{f(t)\} = \int_{-\infty}^{+\infty} f(t)e^{-j\Omega t}dt, \qquad (1)$$

where $-\infty < \Omega < +\infty$. If $f(t)$ is scaled by a ($a > 0$), its

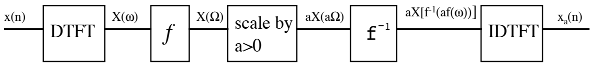

Figure 1: Block diagram of the discrete-time, continuous scaling operator.

Fourier transform becomes

$$\mathcal{F}\{f(t/a)\} = aF(a\Omega),\ -\infty < \Omega < +\infty. \qquad (2)$$

Thus, for a continuous-time signal, a scaling in time can be accomplished in principle by a frequency-scaling of its Fourier transform in the opposite direction along with an amplitude scaling. Now, consider a discrete-time sequence $x(n)$ whose Fourier transform is

$$X(\omega) \equiv \mathcal{G}\{x(n)\} = \sum_n x(n)e^{-j\omega n}. \qquad (3)$$

The function $X(\omega)$ is 2π-periodic. If we try to define a discrete-time scaling operation by adapting (2) to (3), it will only work for integer values of a because of the 2π-periodicity requirement on the Fourier transform of a discrete-time signal. This corresponds to upsampling the discrete-time signal by an integer factor of a. The implementation of our discrete-time continuous scaling operation is as follows (see Figure 1).

1. Given is a discrete-time signal $x(n)$ with (the 2π-periodic) Fourier transform $X(\omega)$.

2. Map the principal interval $\omega \in [-\pi, \pi]$ to continuous frequency Ω (the real line) through an invertible transformation $\Omega = f(\omega)$.

3. Dilate $Y(\Omega) \equiv X(f^{-1}(\Omega))$ by the required dilation factor a to form $Y_a(\Omega) \equiv aY(a\Omega)$.

4. Form $X_a(\omega) = Y_a(f[\omega])$

5. The sequence $x_a(n)$ resulting from the inverse Fourier transformation of $X_a(\omega)$ is the continuous dilation of $x(n)$ by a

$$x_a(n) = a\mathcal{G}^{-1}\{X[f^{-1}(af(\omega))]\}. \qquad (4)$$

where $\mathcal{G}^{-1}$ denotes inverse discrete-time Fourier transform.

Some examples of $f(\omega)$ ($\omega \in [-\pi, \pi]$) are:

- *Bilinear transform.* $\Omega = f(\omega) = 2\tan(\omega/2)$.
- $1/\omega$*-based transform.* $\Omega = f(\omega) = \frac{\omega}{\pi - |\omega|}$.
- *log-based transform.* $\Omega = f(\omega) = sgn(\omega)\ln\left(\frac{\pi}{\pi - |\omega|}\right)$.

2.2. System Properties

Let $\mathcal{S}_a$ denote the discrete-time scaling operator defined above. It is straightforward to verify that $\mathcal{S}_a$ has the following properties:

1. $\mathcal{S}_a$ is a linear operator.

2. $\mathcal{S}_a$ ($a \neq 1$) is a time-varying operator.

3. $\mathcal{S}_1\{x(n)\} = x(n)$ as expected. This corresponds to the non-scaling case.

4. The inverse operator $\mathcal{S}_a^{-1}\{x(n)\} = \mathcal{S}_{1/a}\{x(n)\}$ is discrete-time scaling operation with parameter $1/a$.

5. Commutativity

$$\mathcal{S}_a\{\mathcal{S}_b\{x(n)\}\} = \mathcal{S}_b\{\mathcal{S}_a\{x(n)\}\} = \mathcal{S}_{ab}\{x(n)\} \qquad (5)$$

6. If the discrete-time Fourier spectrum in the principal interval $[-\pi, \pi]$ of an input discrete-time signal is a function of $f(\omega)$, i.e.,

$$X(\omega) = T[f(\omega)], \qquad (6)$$

and the function $T(\omega')$ satisfies

$$T(a\omega') = C(a)T(\omega'), \qquad (7)$$

where C(a) is a function of a, then the output of the discrete-time scaling operator is

$$\mathcal{S}_a\{x(n)\} = aC(a)x(n). \qquad (8)$$

Property 6 provides some interesting insights into the discrete-time scaling operation. It implies that if the inverse Fourier transform of the function $T[f(\omega)]$ exists, the corresponding time sequence represents an eigen-function of the system. Also, when the input spectrum satisfies (6) and (7), for example,

$$T(\omega') = \omega'^r \text{ and hence } X(\omega) = T[f(\omega)] = [f(\omega)]^r, \qquad (9)$$

the output spectrum is identical to the input within an amplitude factor $aC(a)$ (a^{r+1} in the example). In other words, the signal is identical to a scaled version of itself within an amplitude factor.

3. DISCRETE-TIME SELF-SIMILAR SIGNALS

3.1. Self-Similarity

Two types of self-similar signals will be discussed in this paper: deterministic and stochastic.

Definition: A discrete-time sequence $x(n)$ is deterministically self-similar or homogeneous with degree H if it satisfies the following relation.

$$\mathcal{S}_a\{x(n)\} = a^{-H}x(n) \tag{10}$$

for any $a > 0$. A random process $X(n)$ is said to be statistically self-similar with degree H if it satisfies the following equation

$$\mathcal{S}_{a,a}\{R_X(n,n')\} = a^{-2H}R_X(n,n') \tag{11}$$

for any $a > 0$, where $R_X(n,n')$ denotes the auto-correlation function of sequence $X(n)$, and $\mathcal{S}_{a,b}\{x(m,n)\}$ for a 2-D function $x(m,n)$ is defined in lines similar to that of $\mathcal{S}_a$. However, the scaling operation is applied on both m and n dimensions.

3.2. Discrete-Time Homogeneous Signal

As mentioned in section 2.2, the time sequence corresponding to inverse Fourier transform of function $[f(\omega)]^r$, if exists, satisfies (10) with $H = -(r+1)$. Thus, by choosing a function $[f(\omega)]^r$ which is absolutely integrable in $-\pi$ to π, we can derive a class of discrete-time homogeneous functions. This class of homogeneous functions could provide a model for discrete-time self-similar process in practice. They also serve as eigen-functions of the discrete-time scaling operator previously defined.

As we know, the class of continuous-time, regular, homogeneous functions such as $f(t) = 1$ is limited. Truly continuous homogeneous signals corresponding to the spectrum Ω^r do not exist because it is not a valid Fourier spectrum. In our formulation of discrete-time self-similar functions, we are able to derive purely discrete-time sequences as long as $[f(\omega)]^r$ defines a valid discrete-time Fourier spectrum. Non-trivial discrete-time homogeneous functions actually exist and can be derived in the following ranges of r parameter with respect to different mappings.

- Bilinear Transform. $-1 < r < 1$.
- $1/\omega$ Based Transform. $-1 < r < 1$.
- log-Based Transform. $r \neq -1, -2, -3, \ldots$

Figure 2 shows some examples of discrete-time deterministic self-similar functions which are derived from discrete-time Fourier spectrum $[f(\omega)]^{1/2}$. $f(\omega)$ is chosen as bilinear transform, $1/\omega$-based and *log*-based transform respectively.

4. DISCRETE-TIME LINEAR SCALE-INVARIANT SYSTEMS AND SELF-SIMILAR FUNCTIONS

4.1. Discrete-Time Linear Scale-Invariant System

A linear scale-invariant (LSI) system is a linear system whose output is invariant to the scale changes of the input. Let

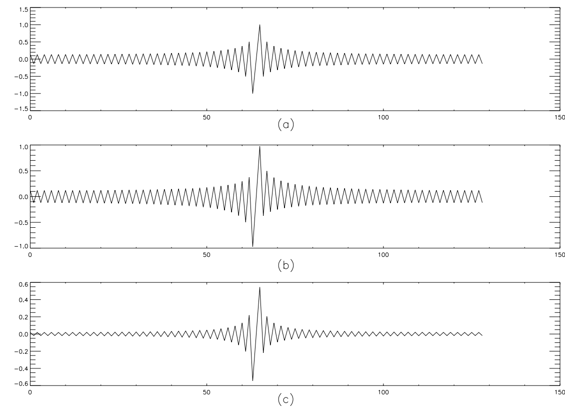

Figure 2: Eigen-functions of the discrete-time scaling operation system with respect to (a) bilinear transform (b) $1/\omega$-based transform (c) *log*-based transform when $r = 0.5$. The imaginary parts of the functions are shown.

$x(n)$, $y(n)$, ($n \in (-\infty, \infty)$) be the input and output sequence. Take an arbitrary discrete-time sequence $h(k)$ ($k = 1, 2, \ldots, K$) and let

$$y(n) = \sum_{k=1}^{K} h(k)\mathcal{S}_k\{x(n)\}/k. \tag{12}$$

This defines a discrete-time LSI system. The output of the system is the summation of a series of dilated (by k) input sequence, weighted by the kernel $h(k)$. Figure 3 shows the procedure for the implementation of the system. Note that the choice of the one dimensional kernel $h(k)$ is arbitrary. Also, as mentioned in section 3, the discrete-time sequences corresponding to spectrum $[f(\omega)]^r$ are eigen-functions of the discrete-time scaling operator. Inherently, they also serve as the eigen-functions of the discrete-time LSI system.

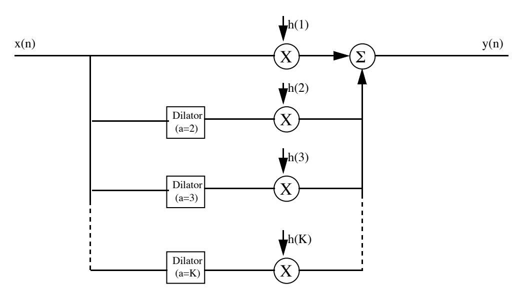

Figure 3: System diagram of the discrete-time LSI system

4.2. Discrete-time Statistically Self-Similar Signal

As mentioned in [1, 3], most physical processes that exhibit statistical self-similarity are fundamentally non-stationary. The statistical properties of the signal change with time, but remain invariant with time scale. In this section we provide a model for such non-stationary self-similar random processes using the discrete-time LSI system. Our implementation in discrete-time domain is based on the following property of the discrete-time LSI system.

Theorem: If the input sequence of a discrete-time LSI system is discrete-time zero-mean white noise, the output sequence of the system is non-stationary and statistically self-similar which satisfies condition (11) with $H = -1$.

proof: See [6].

Hence we can construct a non-stationary self-similar random signal with parameter $H = -1$ by passing a discrete-time zero-mean white noise through the discrete-time LSI system. By passing the signal thus obtained through the system again, a non-stationary self-similar random signal with parameter $H = -2$ is then acquired. Following this scenario, we are able to formulate a non-stationary self-similar random process with parameter H being an arbitrary negative integer. Note that the choice of the one dimensional kernel $h(k)$ in our discrete-time LSI system is essentially arbitrary. We can choose a specific kernel $h(k)$ so that the output of the system exhibits the properties of the studied physical self-similar processes. As there is no restriction on the length of the kernel, a rich class of existing FIR or IIR filters can be applied to model the behavior of a large variety of self-similar random processes in practice.

Figure 4 demonstrates the effect of passing a discrete-time zero-mean white noise through a discrete-time LSI system. The output is a discrete-time stochastic self-similar signal. As is known, if the system output is wide-sense stationary, the 2-D plot of auto-correlation function of the output signal will consist of a series of diagonal straight contour. The auto-correlation plot in Figure 4 clearly demonstrate the non-stationary property of the output signal.

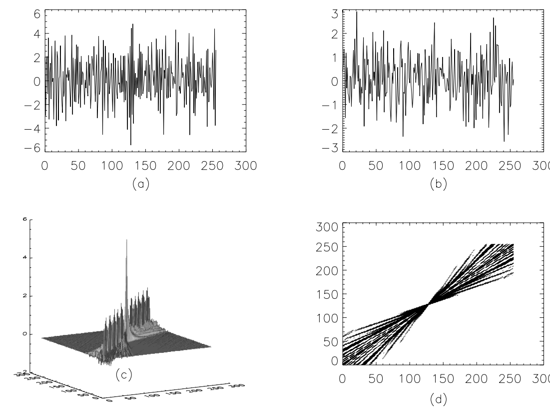

Figure 4: Simulation of passing a zero-mean white noise through discrete-time LSI system. The 6-tap Hanning window is used as 1-d kernel for the discrete-time LSI system. (a) system input (b) system output (c) auto-correlation function of the output (d) contour plot of the auto-correlation function of the output

5. CONCLUSION

In this paper we present a novel model for discrete-time self-similar processes based on a new discrete-time continuous-dilation operation. This model can be viewed as an approach to study the properties of a uniformly sampled self-similar signal. The discrete-time LSI system based on this discrete-time scaling operation provides a potential tool for the analysis and simulation of natural self-similar processes because of its scale-invariant property and flexibility in the choice of one dimensional kernel. It can be emphasized that the study of discrete-time LSI systems deserves further investigation and could contribute to deeper insight into fractals or wavelet applications.

6. REFERENCES

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