

OPTIMAL CUMULANT DOMAIN FILTERING

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ABSTRACT

This paper presents a new technique which exploits constrained optimization methods to derive optimal two dimensional filters in the cumulant domain for processing signals in non Gaussian noise, or signals with corrupting interferences which have non symmetrical probability density functions.

The approach proposed here for enhancing signals in such noise is important, as increasingly practical engineering application areas are identifying occasions where the perceived wisdom of modelling signals in additive Gaussian noise simply does not hold. Since the bispectrum of non Gaussian noise and interference is not zero, it corrupts the bispectrum of the signal. Thus filters that suppress the bispectral component of the noise and enhance the signal bispectrum, are required.

The two dimensional filters proposed in this paper have the property of concentrating the filter energy into a hexagonal region in the bispectral domain. This leads to an impulse response for these filters which represents a new form of two dimensional discrete prolate spheroidal sequence.

The sensitivity of cumulant determination to non Gaussian noise has been noted in the area of array processing [10]. However this paper presents one of the first attempts to remove non Gaussian noise by cumulant filtering.

1. INTRODUCTION

This paper addresses a new problem in Higher Order Statistics (HOS), that of low pass filtering in the two dimensional cumulant domain which exploits third order statistical based algorithms operating on data where the assumption of additive Gaussian noise to a signal does not hold.

The filters presented in this paper concentrate the filter energy into a desired region in the bispectral domain which leads to an impulse response and magnitude squared function for these filters, that represent a new form of two dimensional discrete prolate spheroidal sequence (DPSS) and discrete prolate spheroidal wave function (DPSWF) respectively. The fundamental properties of one dimensional DPSS can be found in [2] and their application to one dimensional finite impulse response (FIR) filters in [3,4]. The extension of these techniques to one dimensional infinite impulse response (IIR) filtering was presented by the authors in [5,6], and extended to two dimensional filters with circularly symmetric passband in [7].

1.1 Problem Definition

Consider a zero mean signal sequence $\{x(n)\}$ whose first three cumulants are defined as [1]:

$$C_1(n) = E[x(n)]$$

$$C_2(n) = E[x(n)x(n+p)] \quad 1$$

$$C_3(p,q) = E[x(n)x(n+p)x(n+q)]$$

where $E[\cdot]$ represents the expectation operator. It is observed that the second order cumulant $C_2(p)$ is simply the autocorrelation sequence of $\{x(n)\}$ and consequently the power spectrum of the sequence is just the Fourier transform of $C_2(p)$. The two dimensional Fourier transform of the third order cumulant

$$S_3(\omega_1, \omega_2) = \sum_p \sum_q C_3(p,q) \exp[-j(p\omega_1 + q\omega_2)] \quad 2$$

is referred to as the bispectrum of the sequence $\{x(n)\}$ [1]. One of the most important properties of HOS is that if $\{x(n)\}$ is a zero mean white Gaussian process, then $C_3(p,q) = 0$ for all values of p and q . This is often exploited in signal processing by assuming that the underlying corrupting noise on a signal is Gaussian, and that consequently the bispectrum of a signal plus noise is simply the bispectrum of the signal.

The fundamental problem addressed in this paper is to accept that in some instances the corrupting noise is non Gaussian and to consider the design of two dimensional digital filters that can be applied to the third order cumulant plane to bandlimit the noise to the bandwidth of the underlying signal. This should enable the bispectrum of a signal to be determined even after it has been corrupted by the bispectrum of non Gaussian noise.

This paper is organized as follows. In Section 2 the philosophy and the design procedure for two dimensional cumulant filters is introduced. Section 3 explains how the impulse response of these filters is a new form of DPSS. Some results are presented for noise corrupted data taken from a rotating shaft, and finally conclusions are presented.

2. CUMULANT FILTER DESIGN

2.1 Design Philosophy

The design philosophy for the new third order cumulant filters follows that published in [3,4,5,6] for one dimensional digital filters based upon DPSS. It was shown in [2] that a one dimensional DPSS is the sequence from all possible sequences

of finite order N , which concentrates the maximum energy into a finite bandwidth $\pm\omega_0$. It is a simple step to take this idea a stage further [3,4] by defining such an N th order DPSS as the impulse response of a one dimensional digital FIR filter, which will then be optimal in the sense that the energy concentration in the filter passband will be a maximum for that order of filter.

In this paper a two dimensional FIR digital filter will be derived which for all such filters of order $N \times N$ turns out to be the one that concentrates the maximum energy into a desired volume in the bispectrum domain: hence the two dimensional impulse response must be a new form of DPSS. The derivation will follow [5,6,7] in that the energy in the filter passband will be maximized using classic Lagrangian multiplier techniques.

2.2 Transfer Function Derivation

The derivation begins by considering a two dimensional non separable FIR filter transfer function of the form:

$$H(z_1, z_2) = \sum_{p=1}^N \sum_{q=1}^N a_{p,q} z_1^{-p} z_2^{-q} \quad 3$$

The frequency response of this transfer function is given by:

$$H(e^{j2\pi f_1}, e^{j2\pi f_2}) = \sum_{p=1}^N \sum_{q=1}^N a_{p,q} e^{-j2\pi f_1 p} e^{-j2\pi f_2 q} = \mathbf{A}^T \mathbf{e} \quad 4$$

where

$$\begin{aligned} \mathbf{e}_1^T &= (e^{-j2\pi f_1} \ e^{-j4\pi f_1} \ \dots \ e^{-j2N\pi f_1}) \\ \mathbf{e}_2^T &= (e^{-j2\pi f_2} \ e^{-j4\pi f_2} \ \dots \ e^{-j2N\pi f_2}) \\ \mathbf{e} &= \mathbf{e}_1 \otimes \mathbf{e}_2 \end{aligned}$$

$\{a_{pq}\}$ = 2-D filter coefficients, and

$$\mathbf{A} = \begin{pmatrix} a_{11} \\ \cdot \\ a_{1N} \\ \cdot \\ a_{NN} \end{pmatrix} \text{ the row ordered equivalent vector.}$$

T represents the transpose operation, \otimes the Kronecker product of two vectors, and it is assumed in 4 that the two sampling frequencies are normalized to unity.

The magnitude squared function for this filter is given by:

$$\left| H(e^{j2\pi f_1}, e^{j2\pi f_2}) \right|^2 = \mathbf{A}^T \mathbf{e} \mathbf{e}^* \mathbf{A} = \mathbf{A}^T \mathbf{Q} \mathbf{A} \quad 5$$

where $*$ represents complex conjugate transpose. The matrix \mathbf{Q} is $N^2 \times N^2$ square matrix whose elements are given by the kernel $\exp[j2\pi(f_1(r-p) + f_2(s-q))]$

The technique employed in this filter design is to maximize the volume (energy) under the magnitude squared function in the

passband, subject to the constraint $\mathbf{A}^T \mathbf{A}$ is a constant. Initially the justification for this constraint is to ensure that the trivial solution that all the coefficients of \mathbf{A} are unity is avoided, but it will be shown later that this constraint also leads to a new form of DPSS with its inherent energy concentrating properties.

The next problem is to define the filter passband in the bispectral domain. It is known that the bispectrum of any lowpass signal will always be concentrated into a hexagonal region in the two dimensional bispectral domain [8], and therefore the passband of a low pass cumulant filter is shown in Figure 1. It is observed from eqn. 5 that the volume under the magnitude squared function in the region of interest is equivalent to evaluating the following integral for the terms in the matrix \mathbf{Q} .

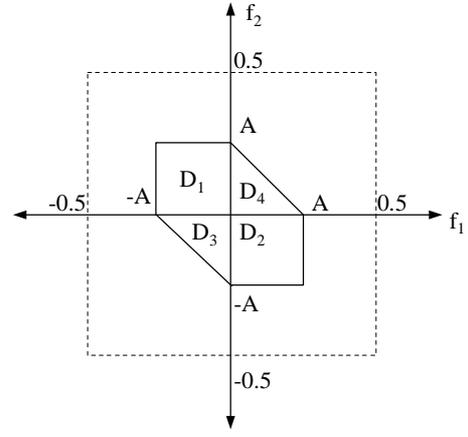


Figure 1. Basic Hexagonal Shape of the Bispectrum of an ideal Bandlimited Signal

$$I = \iint_D \exp[j2\pi(f_1(r-p) + f_2(s-q))] df_1 df_2 \quad 6$$

where D is the hexagonal region in Figure 1. It is observed that the integral in eqn. 6 can be simplified by noting that the hexagonal region D consists of two square regions D_1 and D_2 and two triangular regions D_3 and D_4 . Considering the volume under D_1 first.

$$I_1 = \int_0^A df_2 \int_{-A}^0 \exp[j2\pi(f_1(r-p) + f_2(s-q))] df_1 \quad 7$$

which evaluates to

$$I_1 = A^2 \text{sinc}(\pi A(s-q)) \cdot \text{sinc}(\pi A(r-p)) \cdot \exp(j\pi A[(s-q) - (r-p)]) \quad 8$$

The volume under the square region D_2 can be obtained in a similar manner, and combined with eqn. 8 to yield the volume under the combined square regions D_1 and D_2 .

$$I_{sq} = 2A^2 \text{sinc}(\pi A(r-p)) \cdot \text{sinc}(\pi A(s-q)) \cdot \cos \pi A[(s-q) - (r-p)] \quad 9$$

Now consider the volume under D_4 in Figure 1.

$$I_4 = \iint_{D_4} \exp[j2\pi(f_1(r-p) + f_2(s-q))] df_1 df_2 \quad 10$$

If the following substitution is made

$$\begin{aligned} f_2 &= u \\ f_1 &= v - u \end{aligned}$$

then eqn. 10 can be rewritten as

$$I_4 = \int_0^A dv \int_0^v \exp[j2\pi(r-p)(v-u)] \exp[j2\pi(s-q)u] du \quad 11$$

which evaluates to

$$I_4 = \frac{A}{j2\pi[(s-q) - (r-p)]} \cdot \left[e^{j\pi A(s-q)} \text{sinc}(\pi A(s-q)) - e^{j\pi A(r-p)} \text{sinc}(\pi A(r-p)) \right] \quad 12$$

By obtaining the volume under D_3 in a similar manner, the volume under the triangular regions D_3 and D_4 can be evaluated to be

$$I_{tri} = \frac{A^2}{[(s-q) - (r-p)]} \cdot \left[(s-q) \text{sinc}^2(\pi A(s-q)) - (r-p) \text{sinc}^2(\pi A(r-p)) \right] \quad 13$$

The total volume under the hexagonal region in Figure 1 can be obtained from eqns. 9 and 13

$$I = I_{sq} + I_{tri} \quad 14$$

The volume of the magnitude squared function of the two dimensional digital filter under the hexagonal region in the f_1 - f_2 plane in Figure 1 is given by $\mathbf{A}^T \mathbf{P} \mathbf{A}$ where \mathbf{P} is an $N^2 \times N^2$ block Toeplitz matrix with coefficients given by eqns. 9,13,14.

The filter coefficients are obtained by maximizing the energy in the filter passband subject to a quadratic constraint. The cost function becomes

$$J = \mathbf{A}^T \mathbf{P} \mathbf{A} - \lambda (\mathbf{A}^T \mathbf{A} - K) \quad 15$$

where λ is the Lagrangian multiplier. Differentiating eqn. 15 with respect to \mathbf{A} , and setting this derivative equal to zero produces

$$\mathbf{P} \mathbf{A} = \lambda \mathbf{A} \quad 16$$

From eqn. 16 it is observed that the volume under $\mathbf{A}^T \mathbf{P} \mathbf{A}$ in the filter passband is maximized subject to the quadratic constraint $\mathbf{A}^T \mathbf{A} = K$, if the Lagrangian multiplier λ is an eigenvalue of the matrix \mathbf{P} , and the filter coefficient vector \mathbf{A} is an eigenvector of \mathbf{P} . To determine which eigenvalue of \mathbf{P} yields the optimal

solution, substitute eqn. 16 into the expression for the volume under $\mathbf{A}^T \mathbf{P} \mathbf{A}$ in the passband to obtain:

$$L = \mathbf{A}^T \mathbf{P} \mathbf{A} = \lambda \mathbf{A}^T \mathbf{A} \quad 17$$

It is observed that the volume in the filter passband is maximized if \mathbf{A} is an eigenvector associated with the largest eigenvalue of \mathbf{P} .

3. TWO DIMENSIONAL HEXAGONAL DPSWF AND DPSS

Eqn. 17 can be rewritten as

$$\lambda = \frac{\mathbf{A}^T \mathbf{P} \mathbf{A}}{\mathbf{A}^T \mathbf{A}} \quad 18$$

Note that eqn.18 is the ratio of the energy in the two dimensional hexagonal region in the filter passband to the total energy over the whole two dimensional region of support for the filter. Eqn. 18 is also a Rayleigh quotient, and such quotients are maximized when \mathbf{A} is an eigenvector of \mathbf{P} associated with the largest eigenvalue $\lambda = \lambda_{\max}$. This has already been seen to be the case for these two dimensional filters. Therefore these two dimensional filters are optimal in the sense that for all two dimensional non separable filters of order N these are the ones that concentrate the maximum energy into the hexagonal region in the bispectral domain. Therefore using similar arguments and definitions as [2] it is seen that the filter coefficient matrix obtained from \mathbf{A} via eqn 4 is a form of two dimensional discrete prolate spheroidal sequence and the filter frequency response is a scaled discrete prolate spheroidal wave function.

5. RESULTS

As an example consider data obtained from measurements on a rotating shaft of a machine. This data is essentially low pass in character with two dominant frequencies and it's bispectrum is shown in Figure 2. The data was corrupted with noise having an exponential distribution and the resulting bispectrum shown in Figure 3. This corrupted signal was filtered using a two dimensional cumulant filter with $N=3$ and a value of $A=0.25$ (see Figure 1), and the resulting bispectrum of the filtered signal shown in Figure 4. It is observed that the effect of the exponential noise has been considerably reduced.

6. CONCLUSIONS AND APPLICATIONS

This paper has considered the situation in which the corrupting noise on a signal is non Gaussian in character. Examples of non Gaussian noise are occurring in many areas: for example recent studies of flicker noise processes in hydrogenated amorphous silicon [9], extracted the non Gaussian noise component using a 4th order correlation technique which is very similar to many of the concepts used in HOS.

It has already been reported that non Gaussian noise can cause problems when cumulants are used in signal processing applications [10]. In an array processing application in [10] an

additional noise determining transducer was used to cancel noise and interference, but in other circumstances it becomes appropriate to filter noise from signals and it has been shown in this paper that it is possible to consider filtering in the two dimensional third order cumulant domain.

The equivalent to low pass filtering in the discrete time domain is filtering over a hexagonal region in the cumulant domain and the design of such filters has been considered. In particular the maximization of the filter energy in the hexagonal region subject to a quadratic constraint imposed upon a vector derived from the two dimensional filter matrix has been introduced. It has been shown that this method is equivalent to maximizing the energy concentration of such a two dimensional filter into the hexagonal region in the two dimensional bispectral domain, and consequently the filters produced lead to a new form of two dimensional discrete prolate spheroidal sequence.

5. REFERENCES

- [1] Mendel J.M. "Tutorial on Higher Order Statistics (Spectra) in Signal Processing and Systems Theory: Theoretical Results and some Applications". *Proc. IEEE*, 79:278-305, 1991.
- [2] Slepian D. "Prolate Spheroidal Wave Functions, Fourier Analysis and Uncertainty V: the discrete case". *Bell Syst. Tech. Journal*, 57:1371-1430, 1978.
- [3] Tufts D.W., Francis J.T. "Designing Digital Low Pass Filters-Comparison of some methods and criteria". *IEEE Trans. Audio Electroacoust.*, AU18(4):487-494, 1970
- [4] Papoulis A. Bertram M.S "Digital Filtering and Prolate Functions". *IEEE Trans Cct. Theo.*, CT19(6):674-681, 1972.
- [5] Durrani T.S, Chapman R."IIR Digital Filter Design using Constrained Optimisation Techniques". *IEE Electronic Letters*, 16:843-845, 1980.
- [6] Durrani T.S., Chapman R. "Optimal All-Pole Filter Design based upon Discrete Prolate Spheroidal Sequences". *IEEE Trans. Acoust. Speech and Sig. Proc.*, ASSP32(4):716-721, 1984.
- [7] Chapman R., Durrani T.S. "Circularly Symmetric Filter Design using 2D Prolate Spheroidal Sequences". *IEEE International Conference on Acoustics, Speech, and Signal Processing*. San Diego, March 1984, pages 20.3.1-20.3.4.
- [8] Papoulis A. "Probability Random Variables and Stochastic Processes", (3rd Edition) , McGraw-Hill, 1991
- [9] Abkemeier K.M "Weakly Non Gaussian Processes in a Si:H Conductance Noise", *Phy. Rev B*, 55, 7005, 1997
- [10] Dogan M.C, Mendel J.M "Applications of Cumulants to Array Processing Part II: Non Gaussian Noise Suppression", *USC-SIPI Report #249*, Feb. 1994

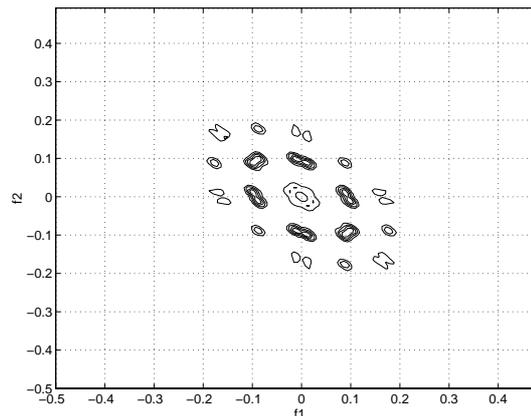


Figure 2: Bispectrum of Original Vibration Data

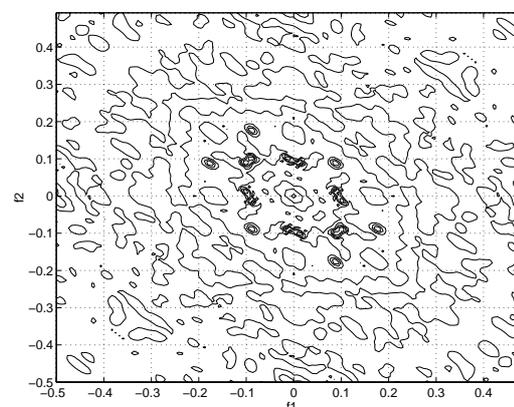


Figure 3 Bispectrum of Original Data Corrupted by Exponential Noise

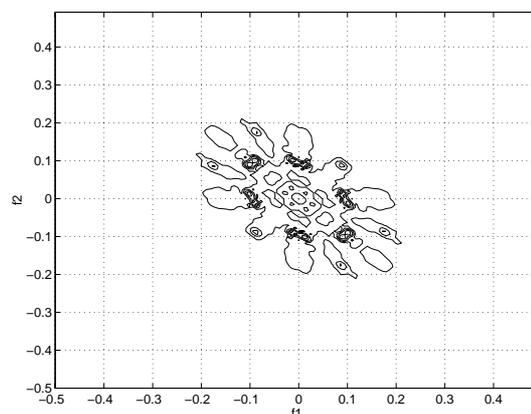


Figure 4 Bispectrum of Corrupted Data filtered by Cumulant Filter: $N=3$, $A=0.25$.