ARBITRARY BANDWIDTH WAVELET SETS

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ABSTRACT

In this paper we consider an extension of the wavelet transform leading to the construction of wavelets with arbitrary bandwidth. The new wavelets are complete, orthonormal and dyadic; nevertheless their bandwidth is not constrained to be one octave, rather it may be designed by selecting a set of parameters. The construction of the new bases starts in the discrete-time domain, exploiting properties of the Laguerre transform. Furthermore, we provide a procedure to define continuous-time warped wavelets. Flexibility of the bandwidth allocation allows for more and improved applications of the wavelet transform, such as signal coding, the design of auditory model based filterbanks and transient detection in pseudoperiodic signals, pointed out in the paper.

1. INTRODUCTION

Dyadic wavelets are a landmark of the wavelet theory since they are the most common and easiest type of wavelets that can be generated starting from a quadrature mirror filterbank (QMF) [8]. However, the octave-band constraint in the frequency resolution of the transform is usually a drawback in the applications. In order to overcome these limitations several approaches were suggested [1, 2, 3], in which the dyadic structure is changed into an M-ary filterbank or rational sampling rate downsamplers are designed. In these approaches, the bandwidth constraints are less severe, although one is not totally free of selecting the appropriate frequency resolution at any given scale level. In several applications, such as analysis, coding and synthesis of audio signals, this is a severe limitation since the relevance of the method and its efficiency are strictly influenced by the freedom of choice of the center-band frequencies or bandwidths.

In this paper we present a class of arbitrary bandwidth wavelets that are based on frequency warping operations via Laguerre transform [11, 13], an instance of unitary equivalence [6]. We consider the iteration of the warped filterbank structure leading to the definition of frequency warped wavelets. At each scale level a different warping map may be selected, thus allowing for a flexible bandwidth design that extends the results found by the authors in [11, 12]. Also, we will show how to derive new continuous-time wavelets from the discrete-time wavelets .

2. WARPED FILTERBANKS

The building block of the frequency warped wavelet transform is the warped filterbank, shown in Fig. 1. The input signal is projected onto a discrete-time Laguerre basis, performed by the Laguerre transform block, and an ordinary QMF critically sampled filterbank with downsampling factor 2 is applied to the sequence of Laguerre coefficients u_r [11]. It can be shown that the coefficient sequence corresponds to a filtered frequency-warped version of the signal $x(k)$, [4, 12]. In fact, in the frequency domain we have:

$$X(e^{j\omega}) = \Lambda_0(e^{j\omega})U(e^{j\theta(\omega)}) \ , \qquad (1)$$

where

$$\Lambda_0(z) = \frac{\sqrt{1-b^2}}{1-bz^{-1}} \ , \qquad (2)$$

and the invertible warping map

$$\theta(\omega) = -\arg A(e^{j\omega}) = \omega + 2\tan^{-1}\left(\frac{b\sin\omega}{1-b\cos\omega}\right) \qquad (3)$$

is the phase response of a first-order real all-pass $A(z)$. The Laguerre transform depends on a parameter b, which controls the amount of warping. The Laguerre transform block may be implemented in a chain of first order IIR filters [5]. All the filters in the chain are all-pass with the exception of the first filter $\Lambda_0(z)$, which is low-pass or high-pass according to the sign of the Laguerre parameter b. Clearly, passing the Laguerre coefficient sequence through the 2-channel perfect reconstruction filterbank results into two sequences, allowing one to reconstruct the sequence of Laguerre coefficients by means of the inverse filterbank (synthesis).

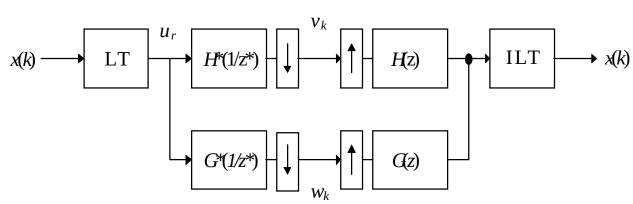

Figure 1. Frequency warped filterbank structure including the Laguerre Transform and Inverse Laguerre Transform blocks.

Finally, the inverse Laguerre block restores the original signal. The frequency characteristics of the warped filterbank can be explained by the following argument. Convolving the warped signal with a QMF response $h(n)$ is equivalent to convolving the signal with an inversely pre-warped version of $h(n)$ and warping the result. The passband of $H(A(z)^{-1})$ is wider or narrower than that of $H(z)$ if, respectively, b is negative or positive. However, as a result of the final warping, the bandwidth of the output signal is at most the same as that of $H(z)$. Therefore, on one hand the cutoff frequency of the warped filterbank may be placed anywhere between 0 and π, on the other hand the final bandwidth of the signal before downsampling is half-band and a downsampling factor 2 may be applied. An example of warped QMF transfer functions is given in Fig. 2.

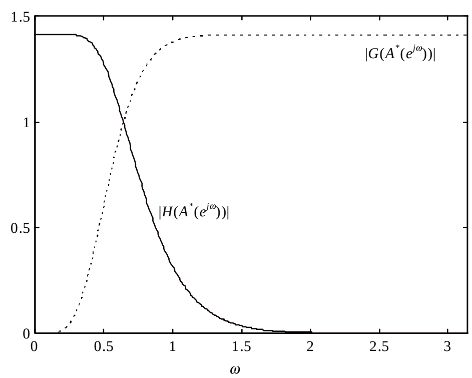

Figure 2. Frequency warped QMF transfer functions

The frequency warped filterbank is a perfect reconstruction structure that can be employed in orthogonal or biorthogonal transforms.

3. WARPED WAVELETS

3.1 Discrete-Time Wavelets

Arbitrary bandwidth wavelets may be generated by iterating the frequency warped filterbank structure illustrated in the previous section. The scheme is similar to that of ordinary wavelets except that at each scale level a warped filterbank is employed. It is easy to show that the z-transforms of the warped scaling sequences and wavelets, respectively, have the following form:

$$\Phi_{n,m}(z) = B_n(z)^{2m}\Phi_{n,0}(z) \tag{4}$$

and

$$\Psi_{n,m}(z) = B_n(z)^{2m}\Psi_{n,0}(z) \,, \tag{5}$$

where

$$B_n(z) = A_n(B_{n-1}(z)^{-2}) = A_n(A_{n-1}(\dots A_1(z)^{-2}\dots)^{-2}) \,, \tag{6}$$

with $B_0(z) = z^{-1/2}$, and $A_n(z)$ is the *n-th* scale level first-order all-pass with parameter b_n.

Furthermore,

$$\Phi_{n,0}(z) = \prod_{k=1}^{n}\left\{\Lambda_{k,0}(B_{k-1}(z)^{-2})H(B_k(z)^{-1})\right\} \tag{7}$$

and

$$\Psi_{n,0}(z) = \Lambda_{n,0}(B_{n-1}(z)^{-2})G(B_n(z)^{-1})\Phi_{n-1,0}(z) \;. \tag{8}$$

The allpass $B_n(z)$ determines the total amount of warping of the *n-th* stage QMFs. The warping map obtained from the phase of the $B_n(z)$ is given by the composite function

$$\Omega_n(\omega) = \theta_n(2\theta_{n-1}(\dots 2\theta_2(2\theta_1(\omega))\dots)) \;, \tag{9}$$

where each $\theta_k(\omega)$ has the form (3), with parameter b_k. By selecting these parameters, the cutoff frequencies ω_k of the warped filters, subject to the constraint $\omega_1 > \omega_2 > \dots > \omega_N$, may be chosen arbitrarily, depending on the application. The cutoff frequency of the ordinary QMFs at $\omega = \frac{\pi}{2}$ transforms into the smallest positive root of the equation $\Omega_n(\omega) = \frac{\pi}{2}$, which is the cutoff frequency of the warped filters. Since the map $\theta^{-1}(\omega)$ is obtained from (3) by reversing the sign of the parameter b, given an admissible choice of cutoff frequencies the parameters b_k may be determined iteratively as follows :

$$b_1 = \tan\left(\frac{\pi - 2\omega_1}{4}\right),$$

$$b_n = \tan\left(\frac{\pi}{4} - \Omega_{n-1}(\omega_n)\right), \quad n=2,3,\dots \tag{10}$$

A typical choice of cutoff frequencies, derived from the structure of classical wavelet transforms, is $\omega_n = \pi a^n$, where a<1. In fig. 3 a set of discrete-time frequency warped wavelets with $a = 2/3$ is shown.

Unlike ordinary wavelets, the warped wavelets are shift-variant. This is due to the replacement of the simple delay term with all-pass filters in (4) and (5). However, at each scale level their magnitude FT is independent of the "shift" index *m*. The tiling of the time-frequency plane is characterized by constant area curvilinear cells delimited by the frequency dependent phase delay portraits, shown in fig. 4.

3.2 Continuous-Time Wavelets

In order to define continuous-time warped wavelets from their discrete-time counterpart detailed in the previous section, one needs to map the analog frequency into the digital frequency. In ordinary wavelets this is simply obtained by rescaling the analog frequency by a factor 2^N at the level N approximation. In the warped wavelet case one needs to consider the map $\Omega_n^{-1}(\omega/2)$, mapping the interval $[-2^n\pi,\, 2^n\pi)$ -- the analog frequency -- onto $[-\pi,\pi)$ -- the digital frequency. Accordingly, we define the N^{th} scaling function approximant as follows:

$$\tilde{\Phi}^{(N)}(\omega) \equiv C^N \Phi_{N,0}\left(e^{j\Omega_N^{-1}(\omega/2)}\right) , \qquad (11)$$

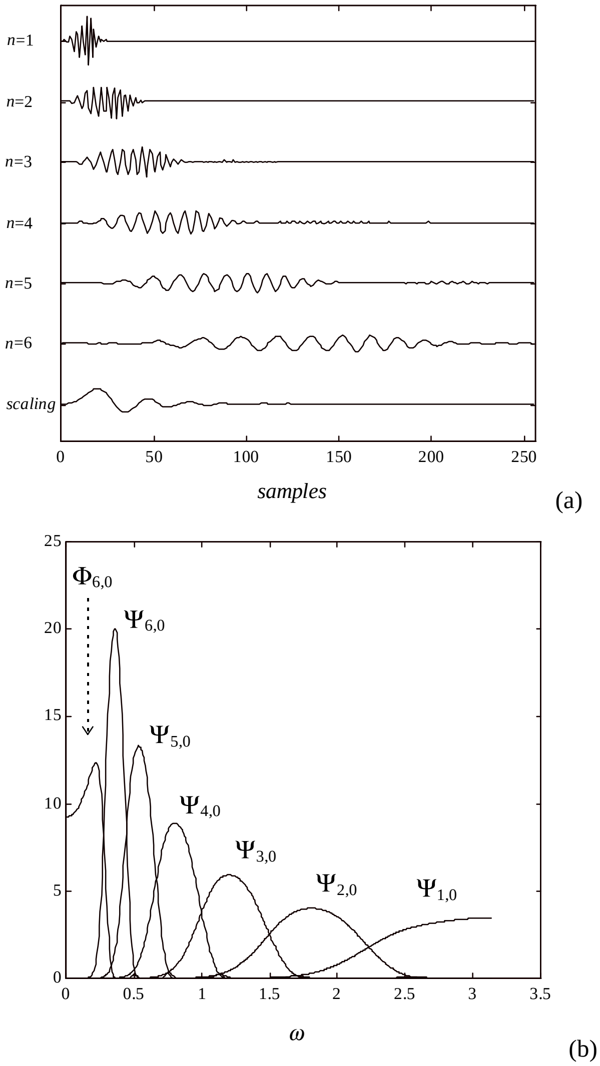

Figure 3. Warped wavelets with exponential cutoff, a=2/3: (a) time domain and (b) frequency domain.

where $\omega \in [-2^n \pi,\ 2^n \pi)$ and C is a constant to be determined. By substituting the product (7) in (11), we obtain:

$$\tilde{\Phi}^{(N)}(\omega) \ = \prod_{k=1}^{N} \left\{ C\Lambda_{k,0}(e^{j2P_{k-1}(\omega)}) H(e^{jP_k(\omega)}) \right\}, \qquad (12)$$

where

$$P_k(\omega) = \Omega_k(\Omega_N^{-1}(\omega/2)) \ .$$

For a constant parameter sequence $b_k = b$, $k=1,2,\ldots$, eq. (12) may be simplified since in this case: $P_k(\omega) = 2^{-1}\Omega_{N-k}(\omega/2)$,

obtaining

$$\tilde{\Phi}^{(N)}(\omega) = \prod_{l=0}^{N-1} \left\{ C\Lambda_0(e^{j\Omega_{l+1}^{-1}(\omega/2)}) H(e^{j2^{-1}\Omega_l^{-1}(\omega/2)}) \right\} \ . \qquad (13)$$

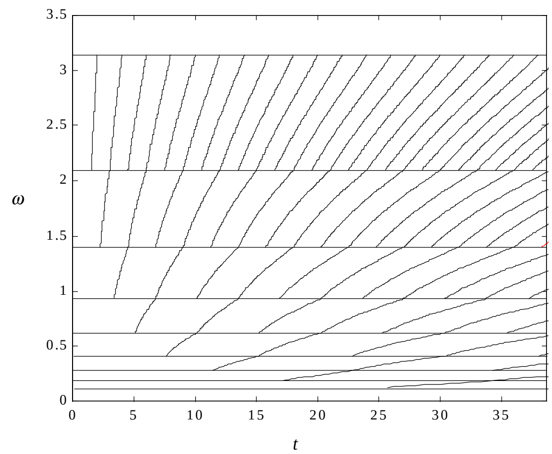

Figure 4. Tiling the time-frequency plane with frequency warped wavelets.

The warped scaling function is defined as the limit of the N^{th} scaling function approximant:

$$\tilde{\Phi}(\omega) \equiv \lim_{N} \tilde{\Phi}^{(N)}(\omega) \quad , \qquad (14)$$

if the infinite product (13) converges. The constant C can be determined from the following relationship:

$$\frac{1}{C} = \Lambda_0(1)H(1) = \sqrt{2\frac{1+b}{1-b}} \ ,$$

derived from a necessary condition for convergence, i.e., by forcing the factors to converge to 1 as $l \to \infty$. From (13) and (14) one can derive the warped form of the two-scale equation:

$$\tilde{\Phi}(2\Omega_1(\omega)) = C\Lambda_0(e^{j\omega}) H(e^{j\Omega_1(\omega)}) \tilde{\Phi}(\omega) \ . \qquad (15)$$

From (15) one arrives at the following definition of frequency warped scaling functions and wavelets at arbitrary scale level:

$$\begin{aligned} &\tilde{\Phi}_{n,0}(\omega) = C^{-n}\tilde{\Phi}(2\Omega_n(\omega)) = \Phi_{n,0}(\omega)\tilde{\Phi}(\omega) \\ &\tilde{\Phi}_{n,m}(\omega) = e^{-j2m\Omega_n(\omega)}\tilde{\Phi}_{n,0}(\omega) \\ &\tilde{\Psi}_{n,0}(\omega) = \Psi_{n,0}(\omega)\tilde{\Phi}(\omega) \\ &\tilde{\Psi}_{n,m}(\omega) = e^{-j2m\Omega_n(\omega)}\tilde{\Psi}_{n,0}(\omega) \ . \end{aligned} \qquad (16)$$

The exponential cutoff choice $\omega_n = \pi a^n$ may be well approximated by the constant b_k case. One can show that the sequence of parameters converges exponentially to the limit value $b = \frac{1-2a}{1+2a}$. Since eq. (13) is not valid in this case, the scaling function approximant will keep the general form (12). However, due to the fast convergence of the parameter sequence it is always possible to split the product (12) into two products. The product corresponding to large values of the index behaves approximately as in the constant *b* case. One can see that the iterated warping map is selfsimilar. Wavelets at larger scales are approximately obtained by scaling the wavelets at smaller scale by a power of *a*.

4. APPLICATIONS

The frequency warped wavelets can be adapted to signal features, e.g., in order to separate them by projection on distinct basis sequences. In audio analysis, synthesis and coding one can optimize the bandwidth allocation of the channels by searching in the Laguerre parameter space.

In coding, one is interested in equalizing the quantization error variances; this is usually achieved by allocating an optimal number of bits per channel [10]. However, the number of bits must be rounded to an integer number of bits per sample. If one can reallocate the bandwidths, the coding gain may be improved if the required floating bit rates get closer to integer bit rates.

Another application of the frequency warped wavelets is in the design of auditory model based filterbanks. In this case one is interested in realizing a filterbank whose channel bandwidths are adapted according to a perceptual criterion [15], e.g., as in mel scale. In this field, the warped wavelets provide a more accurate scheme than the early wavelet transforms, which provided octave band resolution. Furthermore, by choosing the cutoff frequencies, one can obtain non-uniform bandwidths at higher frequencies and uniform bandwidths at lower frequencies, as required in order to approximate the sensitivity of the human ear [16].

The frequency warping scheme may be embedded in other forms of wavelet transform or wavelet packets. An interesting application arises when warping is applied to the pitch-synchronous wavelet transform [9]. This transform allows for separation of pseudo-periodic signals in terms of transients or noise components and an harmonic trend, each corresponding to a different scale level. However, the definition of the transform limited its application to signals whose partials are equally spaced. A large class of audio signals, such as recorded piano tones in the low register, drums, and, more generally, stationary waves propagating in a dispersive medium, show a quasi-harmonic structure in which the spacing of the partials increase with their order [17]. In this case one can perform frequency warping prior to PS-wavelet transform. By embedding this operation in the transform one arrives at the definition of pitch-synchronous frequency warped wavelet transform, useful for performing feature extraction, denoising and excitation modeling of pseudo-periodic signal with nonuniform spacing of partials [14].

The new flexibility is achieved at the cost of increased computational complexity, due to the iterated all-pass filtering required by the Laguerre transform [7, 4]. However, in many applications, the improved frequency resolution may be the main issue.

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