

# PILOT SYMBOL ASSISTED DIVERSITY RECEPTION FOR A FADING CHANNEL

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## ABSTRACT

Pilot symbol assisted modulation is a promising scheme to mitigate the effect of fading in a wireless channel. Analytical results for the performance of this scheme are available. Although the use of diversity is known to improve the performance of receivers used in fading channels, pilot symbol assisted diversity reception has not been studied. In this paper, we derive an exact probability of error expression for such a receiver as a function of the channel estimation error variance and the number of diversity channels. An upper bound for the probability of error, which illustrates the advantage of using diversity, is also obtained. A numerical example is provided.

## 1. INTRODUCTION

A fading channel introduces a random, time-varying amplitude gain and a phase shift to a signal. This gain and phase shift must be estimated in order to perform a coherent detection of a phase modulated signal. In previous work, the use of pilot symbols to estimate these random parameters has been studied. In this scheme, known symbols (pilot symbols) are periodically inserted into the data stream. At the receiver, the samples corresponding to pilot symbol positions are used to estimate the unknown phase and gain at the data symbol positions. Analytical results for this scheme are available in the literature [1].

It is known that diversity reception significantly improves the performance of detection schemes used in fading channels. However, the use of diversity in a pilot symbol assisted scheme has not been studied, to the best of our knowledge. Therefore, in this work, we consider the use of diversity reception in a pilot symbol assisted modulation scheme. We assume that the diversity receiver has  $L$  branches with each branch receiving the signal through an independently fading channel. At each branch, the pilot symbols are used to estimate the phase and gain. These estimates are then combined using *maximal ratio combining* rule to obtain a final

decision on the data bits. We derive an exact probability of error expression for this scheme and show that the error rate decreases exponentially fast with diversity.

The rest of the paper is organized as follows. In section 2, we give a brief description of a fading channel. In section 3, we describe the pilot symbol assisted modulation (PSAM) scheme. In section 4, we describe the diversity receiver and obtain probability of error expressions. A numerical example is given in section 5. Finally, section 6 concludes this paper.

## 2. THE FADING CHANNEL

We consider the frequency non-selective fading channel. The received signal is matched filtered and sampled at the symbol rate. The sample for the  $k^{th}$  symbol can be represented as

$$y(k) = \sqrt{E} c(k) b(k) + n(k) \quad (1)$$

where  $E$  is received energy,  $c(k)$  is the random amplitude gain and phase shift represented as a complex quantity,  $b(k)$  is the  $k^{th}$  data bit, which is assumed to be equally likely to be  $+1$  or  $-1$  and  $n(k)$  is white, additive, complex Gaussian noise with  $\mathcal{E}\{n(k)n^*(k-m)\} = N_0\delta(m)$ . We call  $\{c(k)\}$  the fading process and  $c(k)$  the channel state at the  $k^{th}$  symbol. The real and imaginary parts of  $\{c(k)\}$  are independent, identically distributed Gaussian random processes. The autocorrelation of  $\{c(k)\}$  is given by

$$\mathcal{E}\{c(k)c^*(k-m)\} = J_0(2\pi f_D T m) \quad (2)$$

where  $J_0(\cdot)$  is the order zero Bessel function,  $f_D$  is the maximum Doppler shift and  $T$  is the symbol duration [2]. This time-correlation (or memory) of the fading process enables us to estimate the channel state at a given position when the channel state is known at some adjacent positions.

## 3. PILOT SYMBOL ASSISTED MODULATION

The transmitted data stream contains pilot symbols at regular intervals. We assume that the value of the pilot symbols

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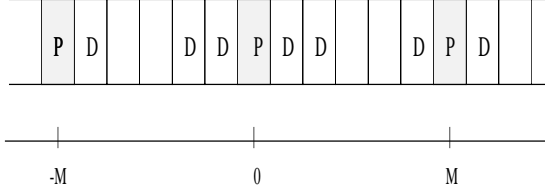


Figure 1: The structure of the data stream – P indicates a pilot position and D indicates a data position.

is  $+1$  and that they are inserted into the data stream at every  $M^{th}$  position. After matched filtering and sampling at the receiver, the sample corresponding to the pilot symbol position is

$$y(iM) = \sqrt{E} c(iM) + n(iM)$$

which is simply the fading process observed in additive noise. Without loss of generality, consider the  $0^{th}$  position to be a pilot symbol so that the transmitted data has the structure shown in Fig. 1. Then the objective is to estimate the unknown channel states  $c(k)$  for the data symbol positions  $k = -\lfloor M/2 \rfloor, \dots, \lfloor M/2 \rfloor$  using the  $K$  nearest pilot symbol positions  $y(-\lfloor K/2 \rfloor M), \dots, y(\lfloor K/2 \rfloor M)$ . A linear estimate for the channel state at the  $k^{th}$  position is obtained as

$$\hat{c}(k) = \sum_{i=-\lfloor K/2 \rfloor}^{\lfloor K/2 \rfloor} w^*(i, k) y(iM).$$

The weights  $w(i, k)$  are selected so that the mean squared error  $\mathcal{E}\{|c(k) - \hat{c}(k)|^2\}$  is minimized. Let

$$\begin{aligned} \mathbf{w}(k) &\triangleq [w(-\lfloor K/2 \rfloor, k) \dots w(\lfloor K/2 \rfloor, k)]', \\ \mathbf{u} &\triangleq [y(-\lfloor K/2 \rfloor M) \dots y(\lfloor K/2 \rfloor M)]'. \end{aligned}$$

Then  $\hat{c}(k) = \mathbf{w}^\dagger(k) \mathbf{u}$  and the MMSE weights are given by

$$\mathbf{R} \mathbf{w}(k) = \mathbf{p}(k)$$

where  $\mathbf{R} = \mathcal{E}\{\mathbf{u} \mathbf{u}^\dagger\}$  and  $\mathbf{p}(k) = \mathcal{E}\{\mathbf{u} c^*(k)\}$  [3]. The MMSE is given by

$$\sigma^2(k) = 1 - \mathbf{p}^\dagger(k) \mathbf{R}^{-1} \mathbf{p}(k). \quad (3)$$

Note that each data position  $k = -\lfloor M/2 \rfloor, \dots, \lfloor M/2 \rfloor$  has a weight vector  $\mathbf{w}(k)$  and a corresponding MMSE  $\sigma^2(k)$ . When an estimate of the channel state is obtained, the transmitted bit in the  $k^{th}$  position is decided as

$$\hat{b}(k) = \text{sgn}\{\Re\{\hat{c}^*(k) y(k)\}\}. \quad (4)$$

The scheme is carried forward by considering the pilot symbols centered at the  $M^{th}$  position and estimating the channel states for the data positions around the  $M^{th}$  position and so forth.

Using the method of Stein [4], as illustrated in [5], the probability of error for this decision scheme can be obtained as

$$\Pr_k(e) = \frac{1}{2} \left[ 1 - \sqrt{\frac{(1 - \sigma^2(k))E/N_0}{1 + E/N_0}} \right] \quad (5)$$

The overall probability of error is the average over the positions, i.e.,

$$\Pr(e) = \frac{1}{M} \sum_{k=-\lfloor M/2 \rfloor, k \neq 0}^{\lfloor M/2 \rfloor} \Pr_k(e). \quad (6)$$

Observe that if the channel state is perfectly known, i.e.,  $\hat{c}(k) = c(k)$ , then the resulting minimum probability of error is

$$\Pr(e)_{min} = \frac{1}{2} \left[ 1 - \sqrt{\frac{E/N_0}{1 + E/N_0}} \right]$$

which is obtained from (6) by setting  $\sigma^2(k) = 0$ .

#### 4. DIVERSITY RECEIVER

In this section we consider a diversity receiver with  $L$  branches. The following standard assumptions on diversity channels [6] are made.

1. The  $L$  branches receive the transmission from  $L$  diversity channels. The fading processes among the diversity channels are mutually independent and identically distributed.
2. The additive noise processes among the diversity channels are mutually independent and identically distributed.

After matched filtering and sampling, the result for the  $l^{th}$  branch and the  $k^{th}$  symbol is

$$y_l(k) = \sqrt{E} c_l(k) b(k) + n_l(k) \quad l = 1, \dots, L \quad (7)$$

Assume that each branch obtains an estimate of the channel state as  $\hat{c}_l(k)$  for  $l = 1, \dots, L$  using pilot symbols. Then the *maximal ratio combining* rule is used to obtain a decision on the data bit as

$$\hat{b}(k) = \text{sgn} \left\{ \Re \left\{ \sum_{l=1}^L \hat{c}_l^*(k) y_l(k) \right\} \right\}$$

##### 4.1. Performance of the diversity receiver

The decision variable for the diversity receiver is

$$X(k) = \Re \left\{ \sum_{l=1}^L \hat{c}_l^*(k) y_l(k) \right\}.$$

First, we find the probability of error conditioned on the channel estimates  $\hat{c}_l(k)$ ,  $l = 1, \dots, L$  and then average over

the estimates. Without loss of generality, assume that  $b(k) = +1$ . Then the conditional probability of error can be expressed as

$$\Pr(e \mid \hat{c}_1(k) = x_1, \dots, \hat{c}_L(k) = x_L) = \Pr(X(k) < 0 \mid \hat{c}_1(k) = x_1, \dots, \hat{c}_L(k) = x_L) \quad (8)$$

Conditioned on the transmitted bit and the channel estimates,  $X(k)$  is a Gaussian random variable. Therefore, (8) can be evaluated if the mean and variance of  $X(k)$  are known. These can be obtained as  $\sqrt{E} \sum_{l=1}^L |x_l|^2$  and  $\frac{1}{2}(\sigma^2(k)E + N_0) \sum_{l=1}^L |x_l|^2$  respectively. Then the conditional probability of error is given by

$$Q \left( \sqrt{\frac{2E}{\sigma^2(k)E + N_0} \sum_{l=1}^L |x_l|^2} \right)$$

where  $Q(x) = \frac{1}{\sqrt{2\pi}} \int_x^\infty \exp(-t^2/2) dt$ . We define

$$\gamma_b \triangleq \frac{E}{\sigma^2(k)E + N_0} \sum_{l=1}^L |x_l|^2$$

where  $\gamma_b$  can be interpreted as an instantaneous *effective* SNR so that the conditional probability of error is  $Q(\sqrt{2\gamma_b})$ . As the estimates  $x_l$  are Gaussian,  $\gamma_b$  has chi-squared distribution with  $2L$  degrees of freedom and is given by

$$p(\gamma_b) = \frac{\gamma_b^{L-1}}{\bar{\gamma}_c^L (L-1)!} \exp(-\gamma_b/\bar{\gamma}_c)$$

where

$$\bar{\gamma}_c \triangleq \frac{E}{\sigma^2(k)E + N_0} (1 - \sigma^2(k)).$$

Here  $\bar{\gamma}_c$  can be interpreted as the average *effective* SNR of a diversity branch. Then the probability of error can be written as

$$\int_0^\infty p(\gamma_b) Q(\sqrt{2\gamma_b}) d\gamma_b \quad (9)$$

which can be evaluated as

$$\left( \frac{1-\mu}{2} \right)^L \sum_{l=0}^{L-1} \binom{L-1+l}{l} \left( \frac{1+\mu}{2} \right)^l \quad (10)$$

where

$$\mu = \sqrt{\frac{\bar{\gamma}_c}{1 + \bar{\gamma}_c}}$$

(cf.[6, p 723]). The overall probability of error is obtained by averaging over the positions as in (6).

## 4.2. An upper bound

Although the probability of error (10) is exact, the advantage of using diversity branches is not quite apparent. Therefore, we use the bound  $Q(x) < \frac{1}{2} \exp(-x^2/2)$  and re-evaluate (9). The result is given by

$$\Pr(e) < \frac{1}{2(1 + \bar{\gamma}_c)^L}$$

This result indicates that for a fixed number of diversity branches, the probability of error decreases only inversely in SNR. However, for fixed SNR, the probability of error decreases exponentially fast as the number of diversity branches  $L$  is increased.

## 5. NUMERICAL EXAMPLE

We consider a diversity receiver with the following parameters.

Normalized fading rate, $f_D T$	0.1
Pilot symbol spacing, $M$	7
Number of pilots used in estimation, $K$	11

The probability of error curves for a single branch and two branches are shown by the solid line in Fig. 2. The dash-dash line shows the probability error if the channel state is known perfectly at the receiver. This is the best performance achievable over the fading channel for uncoded transmissions. Note that the performance of pilot symbol assisted modulation is within 1 dB of this best possible performance. Moreover, the advantage of using diversity is clearly evident in this graph.

## 6. CONCLUSION

We considered diversity reception with pilot symbol assisted modulation and derived an exact probability of error expression as a function of the number of diversity branches and the channel estimation error variance. With the aid of a numerical example, we showed that pilot symbol assisted channel estimation in the diversity receiver achieves performance close to what is achievable if the channel state were perfectly known. Additionally, we showed that the receiver performance improves exponentially fast with the number of diversity branches.

## 7. REFERENCES

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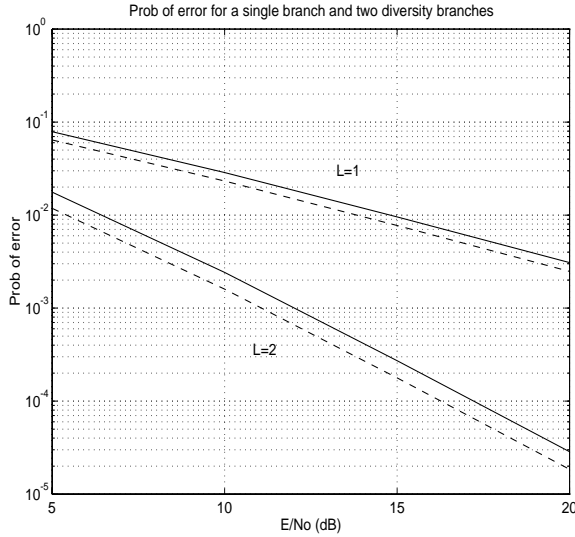


Figure 2: Probability of error for  $L=1$  (single branch) and  $L=2$  (two diversity branches)

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