DESIGNING FRAMES FOR MATCHING PURSUIT ALGORITHMS

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ABSTRACT

A technique for designing frames to use with vector selection algorithms, for example Matching Pursuits (MP), is presented. The design algorithm is iterative and requires a training set of signal vectors. An MP algorithm chooses frame vectors to approximate each training vector. Each vector in the frame is then adjusted by using the residuals for the training vectors which used that particular frame vector in their expansion. The frame design algorithm is applied to speech and electrocardiogram (ECG) signals, and the designed frames are tested on signals outside the training sets. Experiments demonstrate that the approximation capabilities, in terms of mean square error (MSE), of the optimized frames are significantly better than those found using frames designed by ad-hoc techniques. Experiments show typical reduction in MSE by $20 - 50\%$.

1. INTRODUCTION

The goal in traditional transform coding is to represent as much signal information with as few transform coefficients as possible using an orthogonal basis. The optimal transform for a signal depends on the statistics of the stochastic process that produced the signal. If the process is Gaussian it is well known that the optimal transform is given by the eigenvectors of the autocorrelation matrix of the stochastic process and it is called the Karhunen-Loève Transform (KLT). If the process is not Gaussian this need not be true, and it is a nontrivial task to find the optimal transform even if the statistics are known [1]. In addition to these difficulties the signal is often non-stationary, and no fixed transform will be optimal in all signal regions. One way to overcome this problem is to use an overcomplete, or redundant, set of vectors. For a finite dimensional space, any finite overcomplete set of vectors which span the space form a frame [2].

The basic idea when using a frame instead of an orthogonal transform is that we have more vectors and thus a better chance of finding a small number of vectors that match the signal vector well. If a frame contains many vectors, and only one vector is used when approximating each signal vector, it will be equivalent to shape-gain vector quantizer. Typically the frame contains a smaller number of vectors, and more than one frame vector can be used when approximating a signal vector. Since the frame may contain linearly dependent vectors, an expansion is no longer unique. In a compression scheme the goal is to use as few vectors as possible to obtain a good approximation of each signal vector. To find the optimal expansion it is necessary to try all possible combinations. If M vectors are to be used in an approximation of a signal vector, and K is the number of vectors in the frame,

$$\begin{pmatrix} K \\ M \end{pmatrix} = \frac{M!}{K!(M-K)!} \tag{1}$$

possible approximations exists, and finding the best requires extensive calculation. Orthogonal Matching Pursuit (OMP) and Matching Pursuit (MP) are greedy algorithms that are suboptimal, but require far less computations. In this paper we have used OMP.

Goyal and Vetterli have worked with frames or overcomplete expansions [3, 4, 2], using different frames, like vectors on the N-dimensional spheres that maximize the minimum Euclidean norm, or corners of the hypercube $[-\frac{1}{\sqrt{N}}, \frac{1}{\sqrt{N}}]^N$. Goodwin [5] have some arguments for using a particular, ad-hoc based frame. Berg and Mikhael use a frame that contains both the DCT (Discrete Cosine Transform) and the Haar transform for compression of speech signals [6] and images [7]. To the authors' knowledge, no work exists on designing frames for a particular class of signals, but several papers have raised the question [3, 2]. Designing frames using a training set of signal vectors is the topic of this paper.

2. FRAMES AND OMP

A family of functions $\{\varphi_k\}_{k\in K}$ in a Hilbert space H, where K is a countable index set, is called a *frame* if there exist an $A > 0$ and a $B < \infty$, such that for all x in H:

$$A\|x\|^2 \leq \sum_k |\langle \varphi_k, x\rangle|^2 \leq B\|x\|^2. \tag{2}$$

A and B are called *frame bounds*. Let $\mathbf{F}$ denote an $N \times K$ matrix whose columns, $\{\mathbf{f}_j\}$, $j = 0, 1, \ldots, K-1$, constitute a frame. Let $\mathbf{x}_i$ be a real signal vector, $\mathbf{x}_i \in R^N$. $\mathbf{x}_i$ can be approximated as

$$\tilde{\mathbf{x}}_i = \sum_{j=0}^{K-1} w_i(j)\mathbf{f}_j, \tag{3}$$

where $w_i(j)$ is the approximation coefficient corresponding to vector $\mathbf{f}_j$. The corresponding error energy is:

$$\|\mathbf{r}_i\|^2 = \|\mathbf{x}_i - \tilde{\mathbf{x}}_i\|^2 \tag{4}$$

where $\|\cdot\|$ denotes the Euclidean norm in R^N. For a set of M signal vectors, the mean square error (MSE) can be calculated:

$$MSE = \frac{1}{NM} \cdot \sum_{i=0}^{M-1} \|\mathbf{r}_i\|^2. \tag{5}$$

We need to select the frame vectors to be used for approximating a given signal vector $\mathbf{x}$. Orthogonal Matching Pursuit (OMP) is a greedy algorithm for choosing vectors from a frame or dictionary [8]. At each step the OMP selects the vector with the greatest contribution to the gradient of the error energy. In OMP a new vector is calculated as the part of the frame vector that is orthogonal to the vectors that are already chosen, and the corresponding coefficient is calculated for the new vector using the inner product between the new vector and the residual vector.

The Matching Pursuit (MP) algorithm differs from OMP by not finding an orthogonalized new vector, but simply using the selected frame vector as it is and leting the coefficient be the inner product between the selected vector and the residual, thus the OMP gives a better approximation. We use OMP as the vector selection algorithm in this paper, but the design scheme can also be used with MP or other selection algorithms.

3. AN ALGORITHM FOR FRAME DESIGN

The iterative algorithm used to design frames is inspired by the Generalized Lloyd algorithm (GLA) [9]. The GLA requires that each new frame performs better, in terms of MSE, than the previous one using the existing classification. In this context classification includes which frame vectors used in approximating a signal vector *and* the associated coefficients. GLA also requires the frame to perform better after a re-classification, i.e., finding new approximation vectors and coefficients using the new frame. Then the new frame will always be better, or as good as the previous one, and the algorithm at least guarantees a local optimum. There are two problems in using GLA in the context of overcomplete set of vectors. Only one frame vector can be adjusted in each iteration in order to guarantee the new frame to be better using the existing classification, and the selection algorithms used (MP algorithms) are suboptimal. This means that we can not guarantee improvement when re-classifying after an iteration, even if only one vector is adjusted. There is no guarantee for the frame after an iteration to be better than the previous frame. Thus the algorithm presented here is not a GLA, but it is an iterative algorithm similar to GLA.

The main steps in the algorithm are as follows:

1. Begin with an initial frame $\mathbf{F}_0$, $i = 0$.
2. Approximate each training vector according to Equation 3 using a vector selection algorithm. Find the residuals.
3. Given the classifications and residuals, adjust the frame vectors $\Rightarrow \mathbf{F}_i$.
4. Find the new approximations, i.e., re-classify, and calculate the new residuals. If not (stop-criterion = TRUE) $\Rightarrow i = i+1$, go to step 3.

The frame performance is improved by modifying each frame vector according to the residuals from the training vectors that used the actual frame vector in their approximations. Consider a scheme where two frame vectors are selected for approximating each training vector, i.e.:

$$\tilde{\mathbf{x}}_i = w_i(1)\mathbf{f}_{i_1} + w_i(2)\mathbf{f}_{i_2} \tag{6}$$

$$\mathbf{r}_i = \mathbf{x}_i - \tilde{\mathbf{x}}_i, \tag{7}$$

where $\mathbf{x}_i$ is a training vector, $\tilde{\mathbf{x}}_i$ its approximation and $\mathbf{r}_i$ the residual of $\mathbf{x}_i$. Frame vector j is adjusted as

$$\tilde{\mathbf{f}}_j = \mathbf{f}_j + \delta \sum_{k \in T_j} \mathbf{r}_k, \tag{8}$$

where T_j is the set of all training vectors that used $\mathbf{f}_j$ in their expansions. If only one of the frame vectors is adjusted, the new residual for a training vector using this frame vector is

$$\mathbf{r}'_i = \mathbf{r}_i - w_i(j_i)\delta \sum_{k \in T_j} \mathbf{r}_k, \tag{9}$$

where $w_i(j_i)$ is the coefficient corresponding to the adjusted vector, $\mathbf{f}_j$, for the approximation of training vector $\mathbf{x}_i$ before the adjustment of $\mathbf{f}_j$. The total residual of all the training vectors using the frame vector that has been adjusted is:

$$\begin{aligned} \sum_{l \in T_j} \mathbf{r}'_l &= \sum_{l \in T_j} \mathbf{r}_l - \delta \sum_{l \in T_j} w_l(j_l) \sum_{m \in T_j} \mathbf{r}_m \\ &= \sum_{l \in T_j} \mathbf{r}_l - \delta K \sum_{l \in T_j} \mathbf{r}_l \\ &= (1 - \delta K) \sum_{l \in T_j} \mathbf{r}_l \end{aligned} \tag{10}$$

where K is the sum of all the coefficient used with frame vector $\mathbf{f}_j$, before the adjustment:

$$K = \sum_{l \in T_j} w_l(j_l). \tag{11}$$

The residuals for the rest of the training vectors are not influenced by adjustment of the frame vector. From Equation 10 it is seen that if $\| \sum_i \mathbf{r}'_i \| \leq \| \sum_i \mathbf{r}_i \|$, then

$$0 \leq \delta K \leq 2. \tag{12}$$

This means that:

$$\mathrm{sign}(\delta) = \mathrm{sign}(K). \tag{13}$$

Thus for each frame vector in each iteration sign(δ) is set according to Equation 13. If we had used an optimal selection algorithm and adjusted only one vector in each iteration, and then found new approximations and residuals before proceeding with adjusting the next frame vector, the new frame would always be better than the previous, with respect to MSE. Selection algorithms for frames are not optimal, so there is no way to guarantee a better frame when using a practical selection algorithm. In addition, to make the iterations faster we adjust *all* the frame vectors in each iteration, and we normalize the frame vectors to unit length when making a new frame. The proposed algorithm can not guarantee the new frame to be better than the previous, but the results shown in the next section prove that this scheme works well and produces frames that are optimized for a given class of input data. In summary, the algorithm used in this paper work as follows:

1. Begin with an initial frame $\mathbf{F}_0$, $i = 0$.
2. OMP is used to find an approximation for each training vector, and all the residuals are calculated.
3. All frame vectors are adjusted according to Equation 8 with sign(δ) according to Equation 13. The frame vectors are then normalized to unit length $\Rightarrow \mathbf{F}_i$.
4. OMP is used to find the new approximations and residuals. If not (stop-criterion = TRUE) $\Rightarrow i = i + 1$, go to step 3, Else terminate.

Suggested stop-criteria can be: Maximum number of iterations, almost constant MSE, or that the MSE has been growing for some iterations. Due to the lack of guarantee for the new frame to be better than the previous, the stop-criterion can *not* be the normal stop-criterion for GLA i.e., stop if the change in the MSE from last iteration is small enough. The algorithm should allow the MSE to grow for several iterations without terminating the training. This can be seen from training results in the next section.

4. OPTIMIZING FRAMES FOR ECG SIGNALS AND SPEECH SIGNALS

The frame design algorithm is applied to ECG signals and speech signals. In the following experiments a block size of N of 16 is used. Constructing a frame by using segments of a typical signal was shown in [10] to work quite well. In [2] the possibility of adapting the frame by augmenting it with samples from the source is mentioned, but not tried. When using ECG signals as the training set our initial frame is almost the same as the frame used in [10] and it is constructed ad-hoc. DCT vectors were used, in addition to vectors constructed using typical QRS complexes (heartbeats in a normal sinus rhythm). The frame used here consists of 16 DCT vectors and 44 vectors constructed from a typical ECG signal (normal sinus rhythm). In [6] encouraging results are obtained using a frame with DCT and Haar vectors for speech signals. This frame is used as initial frame in our experiments with speech signals. With the chosen block size, 32 frame vectors result.

When designing a frame the number of frame vectors used to approximate a training vector is held constant. A frame is then optimized to be used with that number of vectors in each approximation. When using a frame to approximate a signal, some signal parts may require more frame vectors in the approximation than others, if the approximation quality is to be constant. This is not a problem because several frames can be designed, one for selecting just one vector, one for selecting two, etc. In a compression scheme all the frames can be used when representing a signal without any extra side information since the number of symbols before an end of block (EOB) symbol will indicate which frame to use in each reconstruction.

In Figure 1 training plots are shown for speech and ECG signals. The ECG signals used are signals from the MIT arrhythmia database [11]. The records are represented with 12 bit per sample, and the sampling frequency is 360 Hz. The ECG signal used for training is MIT100, 0:00 to 5:00 minutes, and the variance of the signal is $1.2 \cdot 10^3$. The speech signals used are recorded at 16 kHz in a room without echo, and downsampled to 8 kHz. 8.75 seconds of speech data is used for training, and the variance of the signal is $1.8 \cdot 10^6$. In the experiments with speech signals, using 1, 2, and 3 frame vectors in each approximation, $|\delta| = 10^{-6}$ (Figure 1 a),b), and c)). When experimenting with ECG signals, using 1, 2, and 3 frame vectors in each approximation, $|\delta| = 10^{-5}$ (Figure 1 e), f), and g)). When using 4 vectors in each approximations, for speech signals $|\delta| = 2 \cdot 10^{-7}$ (Figure 1 d)), for ECG signals $|\delta| = 2 \cdot 10^{-6}$ (Figure 1 h)).

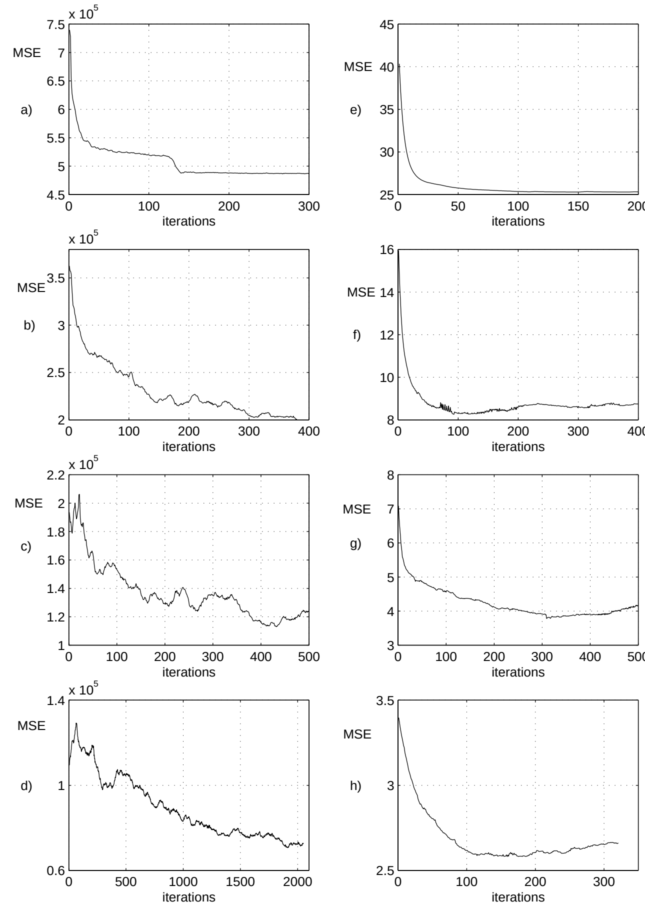

Figure 1: MSE is plotted as a function of training iterations a), b), c), and d) speech signal where 1,2,3, and 4 frame vectors are used in each approximation. e), f), g), and h) ECG signal where 1,2,3, and 4 frame vectors are used in each approximation.

The frames optimized for ECG signals are tested on MIT100, 5:30 to 10:30 minutes, and MIT113, 0:00 to 0:30 minutes. The frames optimized for speech signals are tested on 8.75 seconds of speech recorded under the same conditions as the speech used for training. The optimized frames used for testing are the frames providing the lowest value for MSE during training. In Figure 2 the results using the optimized frames and the initial frames are shown.

5. DISCUSSION AND CONCLUSION

The results demonstrate great potential for optimizing frames for a given class of input data. The comparison in Figure 2 shows that the improvement using the optimized frames, with respect to MSE, is significant. For the experiment with speech signal the re-

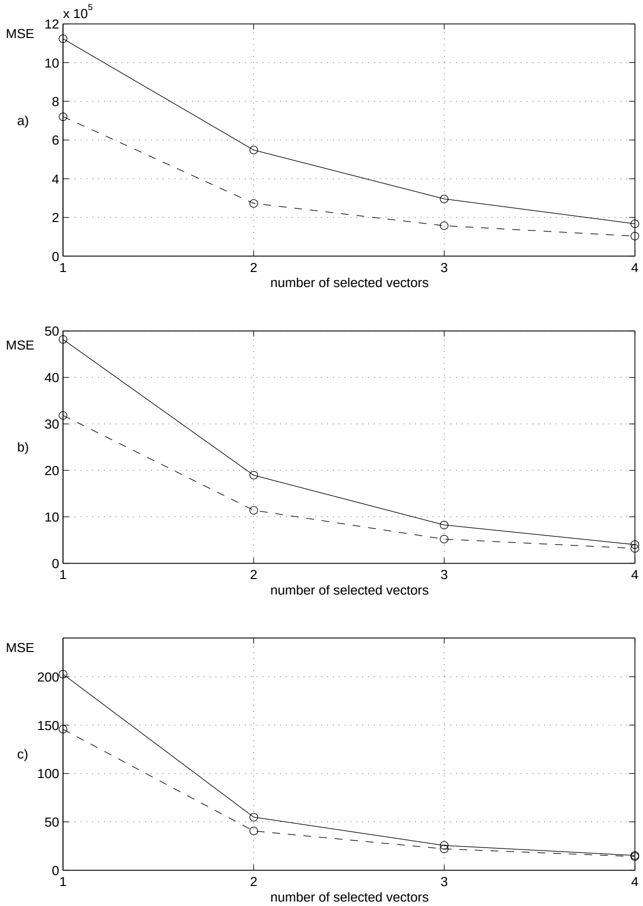

Figure 2: MSE is plotted as a function of different numbers of vectors in an approximation. Test signal is used. a) speech signal, solid: initial frame, dotted: optimized frames b) ECG signal, MIT100, solid: initial frame, dotted: optimized frames, c) ECG signal, MIT113, solid: initial frame, dotted: optimized frames

duction in MSE using the optimized frame compared to the initial frame when using 1,2, 3, and 4 vectors in the approximation are 35.8%, 50.2%, 46.8%, and 32.7%. For the ECG experiments the improvement is largest when using MIT100 as test signal. This is not surprising since the MIT100 test data is data from the same patient as the training set. The MIT113 test signal is also a sinus rhythm, but for another patient. The good results when using few frame vectors in each approximation indicate that this technique will perform well at low bit-rates.

Our experiments demonstrate a gain improvement which is decreasing with the number of vectors used in each approximation. This is intuitively right since when using many vectors in each approximation it is possible to get a good approximation with a lot of different frames.

In the training experiments different δ's had to be used. The difference in δ for the speech signal experiments compared to the ECG signal experiments is probably due to the large difference in the variances of the training signals. Our experiments also indicate that a smaller δ has to be used when using more vectors in each approximation. This may be caused by Equation 12. When more vectors are used in each approximation the variance of K becomes larger. This can cause δ to be out of range for some frame vectors in some iterations, thus a smaller δ may be needed for convergence.

Future work will address the following issues: Convergence, how much the choice of an initial frame influences the design result, and the use of optimized frames in a complete compression scheme for investigating the rate-distortion performance.

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