FACTORIZABILITY OF COMPLEX SIGNALS HIGHER (EVEN) ORDER SPECTRA : A NECESSARY AND SUFFICIENT CONDITION

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ABSTRACT

This paper presents a necessary and sufficient condition for the factorizability of higher order spectra of complex signals. Such a factorizability condition can be used to test if a complex signal can modelize the output of a linear and time invariant system driven by a stationary non gaussian white input. The condition developped here is based on the symmetries of higher order spectra and on an extension of a formula proposed by Marron et al. to unwrap third order spectrum phases. It is an identity between products of six higher order spectra values (which reduces to four values if only phases are considered). Our factorizability test requires no phase unwrapping, unlike existing methods developped in the cepstral domain. Moreover its extension to the N-th order case is direct. Simulations illustrate the deviation to this factorizability condition in a factorizable case (linear system) and a non factorizable case (non linear system).

1. INTRODUCTION

The problem of higher order spectral factorizability has been studied by several researchers. Tekalp and Erdem [13] and Pan and Nikias [10] base their development on higher order cepstrum. Dianat and Raghuveer [6] propose an approach based on MA modelling and Alsheibeili use a LDU decomposition of the cumulant matrix [1]. The three methods compute functions closely related to the higher order spectral factor. For a different purpose, Marron, Sanchez and Sullivan (MSS) have given a formula for phase unwrapping of third order spectra [9]. We show that MSS formula can be used for developping a necessary and sufficient factorizability condition.

In section 2, we generalize MSS formula to the case of higher order spectra of complex signals. Then, we give its expression as an identity between products of higher order spectra. In section 3, the symmetries of higher order spectra and the corresponding matricial operators are given. Then we show in section 4 how this formula associated with higher order spectral symmetries implies the factorizability of the higher order spectrum. Finally, section 5 presents simulation results which show the deviation to the factorizability condition in the case of a linear system and a non linear system.

2. A GENERALIZATION OF MSS FORMULA

Consider the third order spectrum $S_3(u,v)$ of a zero mean non gaussian real random process,

$$S_3(u,v) = E\{X(u)X(v)X(-u-v)\}. \quad (1)$$

When $S_3(u,v)$ is factorizable and if its phase is correctly unwrapped, the following identity is satisfied [9] :

$$\Psi_3(u+w,v) + \Psi_3(u,w) = \Psi_3(u+v,w) + \Psi_3(u,v) \quad (2)$$

where $\Psi_3(u,v)$ is the phase of $S_3(u,v)$. This formula can be generalized as shown below (a detailed development is given in [7].)

We consider complex signals N-th order spectrum defined as the Fourier transform of the N-th order cumulant,

$$S_N(w_1,\ldots,w_{N-1}) = \rho_N(w_1,\ldots,w_{N-1}) e^{j\Psi_N(w_1,\ldots,w_{N-1})}. \quad (3)$$

If factorizability is satisfied, then

$$S_N(w_1,\ldots,w_{N-1}) = \left[\prod_{p=1}^{N/2} H(w_p)\right]\left[\prod_{q=N/2+1}^{N-1} H^*(-w_q)\right] H^*(\sum_{r=1}^{N-1} w_r) \quad (4)$$

where $H(w)$ is the N-th order spectral factor.

When (4) is satisfied, the spectral factor $H(w)$ is unique up to a linear phase term of the form $e^{j(a+bw)}$. Equation (4) generalizes expressions found in [11, 12], as done in [2] with a different choice of variables. The expression of (4) given in terms of phases is

$$\Psi_N(w_1,\ldots,w_{N-1}) = \sum_{p=1}^{N/2} \varphi(w_p) - \sum_{q=N/2+1}^{N-1} \varphi(-w_q) - \varphi(\sum_{r=1}^{N-1} w_r) \quad (5)$$

where $\varphi(w)$ is the phase of $H(w)$.

When (5) holds, the *unwrapped* higher order spectrum phases satisfy :

$$\Psi_N(u+\sum_{p=1}^{N-2} w_p, v_1,\ldots,v_{N-2}) + \Psi_N(u,w_1,\ldots,w_{N-2}) = \Psi_N(u+\sum_{p=1}^{N-2} v_p, w_1,\ldots,w_{N-2}) + \Psi_N(u,v_1,\ldots,v_{N-2}). \quad (6)$$

The result is verified by direct development. A similar equation holds for the higher order spectrum modulus (note the terms in $\log \rho_2$):

$$
\begin{aligned}
&\log \rho_N(u + \sum_{p=1}^{N-2} w_p, v_1, \ldots, v_{N-2}) \\
&+ \log \rho_N(u, w_1, \ldots, w_{N-2}) - \log \rho_2(u + \sum_{p=1}^{N-2} w_p) \\
&= \log \rho_N(u + \sum_{p=1}^{N-2} v_p, w_1, \ldots, w_{N-2}) \\
&+ \log \rho_N(u, v_1, \ldots, v_{N-2}) - \log \rho_2(u + \sum_{p=1}^{N-2} v_p).
\end{aligned} \tag{7}
$$

Consequently, we have the following identity which holds even when the phase is not unwrapped (this identity can be verified by direct development) :

$$
\frac{S_N(u + \sum_{p=1}^{N-2} w_p, v_1, \ldots, v_{N-2}) S_N(u, w_1, \ldots, w_{N-2})}{S_2(u + \sum_{p=1}^{N-2} w_p)} = \frac{S_N(u + \sum_{p=1}^{N-2} v_p, w_1, \ldots, w_{N-2}) S_N(u, v_1, \ldots, v_{N-2})}{S_2(u + \sum_{p=1}^{N-2} v_p)}. \tag{8}
$$

3. SYMMETRIES OF COMPLEX SIGNALS HIGHER ORDER SPECTRA

The factorizability condition is based on MSS formula, but also the symmetries of complex signals higher order spectra. Studies on these symmetries can be found in [3, 4, 5]. Here, we give the matricial operators associed with the transformations of the space that keep the higher order spectrum invariant. We consider only even orders. We have the two types of operations ($M = N/2 - 1$):

- The permutations of the $(M+1)$ first variables or of the M last variables do not modify the N-th order spectrum $S_N(u, v_1, \ldots, v_M, w_1, \ldots, w_M)$. Any of the $(M+1)!M!$ matrices of the form :

$$
P = \begin{bmatrix} Q_{M+1} & 0 \\ 0 & Q_M \end{bmatrix}, \tag{9}
$$

  where Q_{M+1} and Q_M are permutation matrices of dimension $(M+1)$ and M respectively, keep the higher order spectrum invariant.

- The following change of variables:

$$
\begin{aligned}
u' &= u + v_1 + \ldots + v_M + w_1 + \ldots + w_M, \\
v_1' &= -w_1, \ldots, v_M' = -w_M, \\
w_1' &= -v_1, \ldots, w_M' = -v_M,
\end{aligned} \tag{10}
$$

  transforms $S_N(u, v_1, \ldots, v_M, w_1, \ldots, w_M)$ in its complex conjugate. As a consequence, when applied to the variables, the operator T,

$$
T = \left[\begin{array}{c|cccccc|cccccc}
1 & 1 & \cdots & \cdots & \cdots & & 1 & 1 & \cdots & \cdots & \cdots & & 1 \\ \hline
0 & 0 & \cdots & \cdots & \cdots & & 0 & -1 & 0 & \cdots & \cdots & & 0 \\
\vdots & \vdots & \ddots & & & & \vdots & 0 & -1 & \cdots & \cdots & & 0 \\
\vdots & \vdots & & \ddots & & & \vdots & \vdots & \ddots & \ddots & \ddots & & \vdots \\
\vdots & \vdots & & & \ddots & & \vdots & 0 & \cdots & \ddots & -1 & & 0 \\
0 & 0 & \cdots & \cdots & \cdots & & 0 & 0 & \cdots & \cdots & 0 & & -1 \\ \hline
0 & -1 & \cdots & 0 & \cdots & & 0 & 0 & \cdots & \cdots & \cdots & & 0 \\
\vdots & 0 & -1 & \ddots & \cdots & & 0 & \vdots & \ddots & & & & \vdots \\
\vdots & \vdots & \ddots & \ddots & \ddots & & \vdots & \vdots & & \ddots & & & \vdots \\
\vdots & 0 & \cdots & 0 & -1 & & 0 & \vdots & & & \ddots & & \vdots \\
0 & 0 & \cdots & \cdots & 0 & & -1 & 0 & \cdots & \cdots & \cdots & & 0
\end{array}\right] \tag{11}
$$

  keeps the higher order spectrum invariant (if we except the complex conjugation). One can verify that products of matrices of type P and T form a finite group that keep invariant the subspace support of higher order spectra of band limited stationary signals.

4. A NECESSARY AND SUFFICIENT CONDITION FOR FACTORIZABILITY

Theorem : $S_N(u_1, \ldots, u_{N-1})$ is factorizable in the form (4) if and only if (8) is satisfied for a nonzero value of $(v_1, \ldots, v_{N-2})$ and $S_N(u_1, \ldots, u_{N-1})$ satisfies the higher order spectral symmetries of section 3.
The choice of the vector $(v_1, \ldots, v_{N-2})$ requires some care: if $S_N(u_1, \ldots, u_{N-1})$ is periodic, its period must be different from a multiple of $(v_1 + \ldots + v_{N-2})$. If $S_N(u_1, \ldots, u_{N-1})$ is periodic and discrete, which is often the case in applications based on the DFT, it is always possible to choose $(v_1, \ldots, v_{N-2}) = (\Delta\nu, 0, \ldots, 0)$ where $\Delta\nu$ is the frequency resolution.
The necessary condition was shown in sections 2 and 3, the sufficient condition is proven in the next.

4.1. Proof of the condition on the phases when they are correctly unwrapped

If we know a correct (for example the continuous) determination of the higher order spectrum phase, and not only its determination in the range $[-\pi, \pi[$, the factorizability condition in terms of phases is written:

$$
\begin{aligned}
&\Psi_N(u + \sum_{p=1}^{N-2} w_p, v_1, \ldots, v_{N-2}) + \Psi_N(u, w_1, \ldots, w_{N-2}) = \\
&\Psi_N(u + \sum_{p=1}^{N-2} v_p, w_1, \ldots, w_{N-2}) + \Psi_N(u, v_1, \ldots, v_{N-2}).
\end{aligned} \tag{12}
$$

4.1.1. Expression of the higher order spectrum phase as a sum

When $(v_1, \ldots, v_{N-2})$ is fixed, the difference between the two function of $(N-1)$ variables in (12):

$\Psi_N(u+\sum_{p=1}^{N-2} v_p, w_1, \ldots, w_{N-2})$ and $\Psi_N(u, w_1, \ldots, w_{N-2})$, is the difference between two functions of one variable:

$$\Psi_N(u, w_1, \ldots, w_{N-2}) - \Psi_N(u + \sum_{p=1}^{N-2} v_p, w_1, \ldots, w_{N-2}) =$$
$$\Psi_N(u, v_1, \ldots, v_{N-2}) - \Psi_N(u + \sum_{p=1}^{N-2} w_p, v_1, \ldots, v_{N-2}). \tag{13}$$

If $\Psi_N(u, w_1, \ldots, w_{N-2})$ is a periodic function of u, we suppose that we have chosen a vector $(v_1, \ldots, v_{N-2})$ such that $\ell = \sum_{p=1}^{N-2} v_p$ and its multiples are different from its period; otherwise, MSS formula could not be used to check the factorizability. In order to take advantage of this property, we decompose $\Psi_N(u, w_1, \ldots, w_{N-2})$ in a sum:

$$\Psi_N(u, w_1, \ldots, w_{N-2}) =$$
$$f(u) + g(u + \sum_{p=1}^{N-2} w_p) + h(u, w_1, \ldots, w_{N-2}), \tag{14}$$

where $h(u, w_1, \ldots, w_{N-2})$ holds no additive terms depending only on u or $\left(u + \sum_{p=1}^{N-2} w_p\right)$.

4.1.2. $h(u, w_1, \ldots, w_{N-2})$ does not depend on u

Using (14) in computing the difference (13) yields:

$$\Psi_N(u, w_1, \ldots, w_{N-2}) - \Psi_N(u + \ell, w_1, \ldots, w_{N-2}) =$$
$$f(u) + g(u + \sum_{p=1}^{N-2} w_p) + h(u, w_1, \ldots, w_{N-2})$$
$$- f(u+\ell) - g(u + \ell + \sum_{p=1}^{N-2} w_p) - h(u+\ell, w_1, \ldots, w_{N-2}) =$$
$$\Psi_N(u, v_1, \ldots, v_{N-2}) - \Psi_N(u + \sum_{p=1}^{N-2} w_p, v_1, \ldots, v_{N-2}), \tag{15}$$

with $\ell = \sum_{p=1}^{N-2} v_p$. The last line in (15) holds only two terms functions respectively of u and $\left(u + \sum_{p=1}^{N-2} w_p\right)$. Consequently,

$$h(u, w_1, \ldots, w_{N-2}) = h(u+\ell, w_1, \ldots, w_{N-2}). \tag{16}$$

If $(v_1, \ldots, v_{N-2})$ is chosen appropriately, $\Psi_N(u, w_1, \ldots, w_{N-2})$ is not periodic of period ℓ in u and neither is $h(u, w_1, \ldots, w_{N-2})$. So, this last function does not depend on u:

$$h(u, w_1, \ldots, w_{N-2}) = h'(w_1, \ldots, w_{N-2}), \tag{17}$$

and (14) becomes

$$\Psi_N(u, w_1, \ldots, w_{N-2}) =$$
$$f(u) + g(u + \sum_{p=1}^{N-2} w_p) + h'(w_1, \ldots, w_{N-2}). \tag{18}$$

4.1.3. Application of the symmetries

Now we can apply the changes of variables of section 3, that keep the higher order spectrum invariant. First the permutation of u and any of the $(N/2-1)$ variables $w_1, w_2, \ldots, w_{N/2-1}$ yields:

$$h'(w_1, \ldots, w_p, \ldots, w_{N/2-1}, w_{N/2}, \ldots, w_{N-2}) =$$
$$f(w_p) + h'(w_1, \ldots, u, \ldots, w_{N/2-1}, w_{N/2}, \ldots, w_{N-2}) - f(u), \tag{19}$$

so that

$$h'(w_1, \ldots, w_{N/2-1}, w_{N/2}, \ldots, w_{N-2}) =$$
$$\sum_{p=1}^{N/2-1} f(w_p) + h''(w_{N/2}, \ldots, w_{N-2}). \tag{20}$$

If we use the transformation T,

$$h''(w_{N/2}, \ldots, w_{N-2}) = -\sum_{p=N/2}^{N-2} f(-w_p), \tag{21}$$

and

$$g(u) = -f(u). \tag{22}$$

Finally

$$\Psi_N(u, w_1, \ldots, w_{N-2}) =$$
$$f(u) + \sum_{p=1}^{N/2-1} f(w_p) - \sum_{p=N/2}^{N-2} f(-w_p) - f(u + \sum_{p=1}^{N-2} w_p), \tag{23}$$

which expresses the factorizability.

4.2. On the computation of the correct phase determination (phase unwrapping)

If (12) is satisfied, $\Psi_N(u, w_1, \ldots, w_{N-2})$ can be decomposed as in (23) provided that $\Psi_N(u, w_1, \ldots, w_{N-2})$ is a correct determination of the N-th order spectrum phase in the range $[-4\pi, 4\pi[$. Now, we show that it is always possible to compute the correct determination of $\Psi_N(u, w_1, \ldots, w_{N-2})$ from the one that is known $\Psi_N^0(u, w_1, \ldots, w_{N-2})$ (given in the range $[-\pi, \pi[$):

$$\Psi_N(u, w_1, \ldots, w_{N-2}) =$$
$$\Psi_N^0(u, w_1, \ldots, w_{N-2}) + 2\pi m(u, w_1, \ldots, w_{N-2}), \tag{24}$$

where $m(u, w_1, \ldots, w_{N-2})$ takes integer values. We consider the values of $\Psi_N^0(u, v, 0, \ldots, 0)$ (the third order spectrum phase). A subset of these data can be used to reconstruct the spectral factor phase $f(u)$ using a recursive multiresolution algorithm without raising phase unwrapping difficulties [8]. The recursive structure of the algorithm allows, in theory, the reconstruction of $f(u)$ for all u at any frequency resolution. When $f(u)$ is known, in using (5), it is possible to compute the correct determination (24) of $\Psi_N(u, w_1, \ldots, w_{N-2})$ satisfying (6). (Besides, this was the initial object of the MSS recursion). Starting from the HOS data used in this multiresolution method, there is only one manner to reconstruct recursively $\Psi_N(u, w_1, \ldots, w_{N-2})$ satisfying (24). Consequently when (8) is satisfied, it is always possible to reconstruct the determination of the phase allowing the development of section 4.1 since the spectral factor is unique. The knowledge of $\Psi_N^0(u, w_1, \ldots, w_{N-2})$ in the range $[-\pi, \pi[$ is sufficient.

4.3. Factorizability of the higher order spectrum modulus

A similar development applies to the modulus. When the higher order spectrum symmetries and (7) are satisfied, the modulus logarithm can be written in the form

$$\log \rho_N(u, w_1, \ldots, w_{N-2}) = \lambda(u) + \lambda(u + w_1 + \ldots + w_{N-2}) + \lambda(w_1) + \ldots + \lambda(w_{N/2-1}) + \lambda(-w_{N/2}) + \ldots + \lambda(-w_{N-2}). \quad (25)$$

When (7) is satisfied,

$$2\lambda(u) = \log |S_2(u)|, \quad (26)$$

which gives the modulus of the spectral factor. This implies the factorizability of the modulus.
Consequently, if (8) and the higher order spectrum symmetries hold,
$S_N(u, w_1, \ldots, w_{N-2})$ is factorizable.

5. SIMULATION RESULTS

The simulations show the convergence of the factorizability condition as a function of the number of signal samples n used in the estimation of the fourth order spectrum ($N = 4$). We compute the evolution of the deviation $\mathcal{D}_n$ when n increases.

$$\mathcal{D}_n \triangleq [\Psi_4(u + w_1 + w_2, v_1, v_2) + \Psi_4(u, w_1, w_2) - \Psi_4(u + v_1 + v_2, w_1, w_2) + \Psi_4(u, v_1, v_2)]_n \quad (27)$$

In a first experiment, we compute $\mathcal{D}_n^{(1)}$ for a signal $y[k]$, output of a LTI filter whose input, $x[k]$, is a non gaussian IID sequence ($x[k] = \pm 1$):

$$y[k] = x[k] + 0.3\, x[k-1]. \quad (28)$$

In the second experiment, we compute $\mathcal{D}_n^{(2)}$ for a signal $z[k]$ measured at the output of a non-linear system whose input is $y[k]$:

$$z[k] = y^3[k] + y[k-1]. \quad (29)$$

Figure 1 shows that $\mathcal{D}_n^{(1)}$ converges towards zero and that $\mathcal{D}_n^{(2)}$ converges towards a nonzero value.

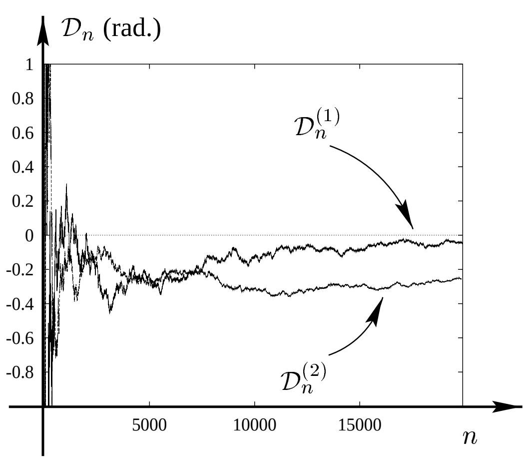

Figure 1: Convergence of the factorizability condition in the linear ($\mathcal{D}_n^{(1)}$) and non-linear case ($\mathcal{D}_n^{(2)}$).

6. CONCLUSION

This paper presents a necessary and sufficient factorizability condition of a N-th order spectrum. Simulation results show the deviation from such a condition in the case of a linear system which satisfy the factorizability condition and a non linear system which does not satisfy this condition.

7. REFERENCES

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