ON THE USE OF A GENERAL AMPLITUDE PDF IN COHERENT DETECTORS OF SIGNALS IN SPHERICALLY INVARIANT INTERFERENCE

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ABSTRACT

The aspects of using a general amplitude probability density function in coherent detectors are investigated. For this purpose, the recently developed Generalised Bessel function K (GBK) distribution is used. The performance of the optimal detector of signals embedded in GBK-distributed interference is compared to the one of the uniformly most powerful invariant detector using extensive Monte Carlo simulations. The results indicate that for small number of integrated pulses the optimal detector outperforms the uniformly most powerful invariant detector by up to 18 dB. It is shown, that this improvement does not vary significantly with changes in the parameters that control the spherical invariance of the GBK distribution.

1. INTRODUCTION

The generalised Bessel function K (GBK) distribution has been introduced recently for modelling physical phenomena such as radar clutter [4].

The GBK distribution is a four-parameter distribution which encompasses a large number of well-known distribution often used in modelling the amplitude pdf of narrow-band random processes such as the Rayleigh, Weibull, and K-distribution.

It has also been established that the design of a coherent detector for deterministic signals with unknown amplitude and phase embedded in GBK-distributed interference can be simplified when the interference is modelled by a spherically invariant random process [5]. Such a detector possesses an important feature of having the same structure for all spherically invariant models included in the GBK distribution. It is also expected that such a detection system will perform better than systems designed for more specific models in situations where the interference statistics change from time to time or from location to location in such an extent that they cannot be modelled by a simpler, say two-parameter, distribution.

The problem of detecting signals in spherically invariant interference can be posed as a problem of detecting signals in a conditionally Gaussian process. Denote a nonnegative random variable by S with probability density function (pdf) $f_S(s)$. Let

$$\tilde{\boldsymbol{Z}} = S\,\tilde{\boldsymbol{X}} = S(\boldsymbol{X}_I + j\boldsymbol{X}_Q) = \boldsymbol{Z}_I + j\boldsymbol{Z}_Q$$

where subscripts $_I$ and $_Q$ refer to the inphase and the quadrature components of the process, respectively, and $\boldsymbol{X} = [\boldsymbol{X}_I, \boldsymbol{X}_Q]$ is a zero-mean Gaussian vector valued random variable independent of S. Consequently, the vector $\boldsymbol{Z} = [\boldsymbol{Z}_I, \boldsymbol{Z}_Q]$ is called a spherically invariant random vector [6, 1].

To solve the problem of detecting signals embedded in $\boldsymbol{Z}$ one may choose to estimate the parameters of the amplitude pdf $f_S(s)$ (if its form is known) and the covariance matrix of the Gaussian process and design an optimal detector in the Neyman-Pearson sense (see, for example [2, 5]).

Another method is to treat S as an unknown (nuisance) parameter and consider the classical solution of a uniform most powerful invariant (UMPI) detector [7]. It has been shown that for large number of integrated pulses (greater than 20), there is no advantage to be gained from using full information on $f_S(s)$ [7, 3].

The paper is organised as follows. In the next section, the multivariate representation of the GBK distribution is presented. In section 3, the design of an optimal detector for signals embedded in GBK-distributed interference is considered. In section 4, the performance of the optimal detector is compared to the one of the UMPI detector for a small number of integrated pulses ranging from 2 to 16. The conclusions are provided in section 5.

2. THE GBK DISTRIBUTION

The multivariate representation of the GBK distribution is given by [4]

$$\begin{aligned} f_{\boldsymbol{Z}}(\boldsymbol{z}) &= \frac{4a^N c^{N-1}}{,\ (\alpha_1),\ (\alpha_2)(2\pi)^N |\boldsymbol{M}|^{\frac{1}{2}}} \\ &\times \left(\sqrt{a}\|\boldsymbol{z}\|\right)^{\frac{c}{2}(\alpha_1+\alpha_2+N-1)-2N} \sum_{k=1}^{N} (-1)^{m+k} \\ &\times \frac{P_{(N,k)} K_{(\alpha_2-\alpha_1-k+1)} \left[2\left(\sqrt{a}\|\boldsymbol{z}\|\right)^{\frac{c}{2}}\right]}{\left(\frac{c}{2}\sqrt{a}\|\boldsymbol{z}\|^{\frac{c}{2}}\right)^{N-k}}, \quad (1) \end{aligned}$$

where

$$\boldsymbol{z} = [z_{I1}\ z_{I2}\ \cdots\ z_{IN}\ z_{Q1}\ z_{Q2}\ \cdots\ z_{QN}]^T$$

is a real vector with $2N$ entries of the observations of the random process vector $\boldsymbol{Z}$, $\alpha_1 > 0$, $\alpha_2 > 0$, and $c > 0$ are the distribution parameters, $K_\nu(\cdot)$ is the modified Bessel function of the second kind of order ν, $\boldsymbol{M}$ is the covariance matrix (assumed to be invertible), $\|\cdot\|$ is the Euclidean norm,

$$a = \frac{,\ \left(\alpha_1 + \frac{2}{c}\right),\ \left(\alpha_2 + \frac{2}{c}\right)}{2,\ (\alpha_1),\ (\alpha_2)},$$

$$m = \begin{cases} 1, & N \text{ odd} \\ 0, & N \text{ even,} \end{cases}$$

and the coefficients $P_{(N,k)}$ are calculated recurrently

$$P_{(N,k)} = P_{(N-1,k)}\, C_{(N,k)} + P_{(N-1,k-1)},$$

with

$$C_{(N,k)} = \begin{cases} 0, & k > N \\ 1, & k = N \\ \frac{c}{2}\alpha_1 - (N-1) + \frac{c(k-1)}{2}, & k < N, \end{cases}$$

$P_{(0,0)} = 1$, $P_{(N,0)} = 0$, and $P_{(0,k)} = 0$.

It was established that the theory of spherically invariant random vectors can be applied when the interference amplitude is modelled by the GBK distribution, having pdf given by

$$f_R(r) = \frac{2c\left(\frac{r}{\beta}\right)^{\frac{c}{2}(\alpha_1+\alpha_2)-1}}{\beta,\ (\alpha_1),\ (\alpha_2)} K_{(\alpha_2-\alpha_1)}\left[2\left(\frac{r}{\beta}\right)^{\frac{c}{2}}\right], \quad (2)$$

with the shape parameter α_1 and the power parameter c, such that

$$C_{(N,k)} \leq 0, \qquad k = 1, \ldots, N-1. \quad (3)$$

The GBK distribution given in (2) includes a large number of well-known statistical distributions such as Rayleigh ($\alpha_1 = 0.5, \alpha_2 = 1, \beta = 2\sigma, c = 4$) , Weibull ($\alpha_1 = 0.5, \alpha_2 = 1, \beta = a2^{1/p}, c = 2p$), K-distribution ($\alpha_1 = 1, \alpha_2 = \nu + 1, \beta = 2a, c = 2$) and many others [4].

3. DESIGN OF A DETECTOR

The problem of detecting deterministic signal $\boldsymbol{s} = Ae^{j\theta}\boldsymbol{v}$ in GBK-distributed interference can be expressed in the following framework

$$\begin{aligned} \mathsf{H}_0 &: \boldsymbol{r} = \boldsymbol{z} \\ \mathsf{H}_1 &: \boldsymbol{r} = \boldsymbol{s} + \boldsymbol{z}, \end{aligned} \quad (4)$$

where H_0 denotes the null hypothesis, H_1 the alternative hypothesis, and where $\boldsymbol{r} = [\boldsymbol{r}_I, \boldsymbol{r}_Q]$, $\boldsymbol{z} = [\boldsymbol{z}_I, \boldsymbol{z}_Q]$, and $\boldsymbol{s} = [\boldsymbol{s}_I, \boldsymbol{s}_Q]$, are real vectors with $2N$ entries representing the observations of the received signal, interference, and deterministic signal, respectively.

One can represent a GBK-distributed variate as conditionally Gaussian with some modulating variate $s > 0$. The so called *characteristic probability density function*, $f_S(s)$, can be found by solving the following integral equation [1]

$$f_{\boldsymbol{Z}}(\boldsymbol{z}) = \frac{(2\pi)^{-N}}{|\boldsymbol{M}|^{\frac{1}{2}}} \int_0^\infty \frac{1}{s^{2N}} \exp\left(-\frac{\|\boldsymbol{z}\|^2}{2s^2}\right) f_S(s) ds, \quad (5)$$

where $f_{\boldsymbol{Z}}(\boldsymbol{z})$ is given in (1) and $C_{(N,k)} \leq 0$. However, no closed form expression for $f_S(s)$ has been found. Nevertheless, the knowledge of $f_S(s)$ is not necessary when designing optimal detection structures.

Since the theory of spherically invariant random vectors can be applied when the clutter amplitude is modelled by the GBK distribution with certain values of the parameters α_1 and c, one can use a whitening transformation without penalty, providing that the covariance matric $\boldsymbol{M}$ is known. This is due to the fact that a spherically invariant random vector is closed under a linear transformation [6]. Whitening the received signal leads to the following framework

$$\begin{aligned} \mathsf{H}_0 &: \boldsymbol{x} = \boldsymbol{n} \\ \mathsf{H}_1 &: \boldsymbol{x} = \boldsymbol{u} + \boldsymbol{n}, \end{aligned} \quad (6)$$

where $\boldsymbol{x}$ is the whitened version of the received signal vector $\boldsymbol{r}$, and $\boldsymbol{n}$ and $\boldsymbol{u}$ represent the whitened versions of the interference vector $\boldsymbol{z}$ and deterministic signal vector $\boldsymbol{s}$, respectively. Note that the whitening transformation does not change the statistical properties of a parametric detector as long as the interfering process is spherically invariant.

Since the interfering process is assumed to be a zero-mean process, the whitened version of the target signal can be expressed as $\boldsymbol{u} = Ae^{j\theta}\boldsymbol{p}$, where $\boldsymbol{p}$, referred to as the signal pattern, is the whitened version of $\boldsymbol{v}$.

Optimum detection, in the Neyman-Pearson sense, of a known signal in GBK-distributed interference, can be achieved using the log-likelihood ratio test (LLRT).

In our case, the LLRT is given by

$$\Lambda(\boldsymbol{x}) = \log\left[\frac{f_{\boldsymbol{X}}(\boldsymbol{x}|\mathsf{H}_1)}{f_{\boldsymbol{X}}(\boldsymbol{x}|\mathsf{H}_0)}\right] = \log\left[\frac{f_{\boldsymbol{N}}(\boldsymbol{x}-\boldsymbol{u})}{f_{\boldsymbol{N}}(\boldsymbol{x})}\right] \underset{H_0}{\overset{H_1}{\gtrless}} T, \tag{7}$$

where $f_{\boldsymbol{N}}(\boldsymbol{x})$ is equivalent in form to $f_{\boldsymbol{Z}}(\boldsymbol{z})$ given in (1) but with a unit diagonal covariance matrix, and T is a suitable threshold that controls the probability of false alarm. Substituting (1) into (7) yields

$$\Lambda(\boldsymbol{x}) = g(\|\boldsymbol{x}-\boldsymbol{u}\|) - g(\|\boldsymbol{x}\|) \underset{H_0}{\overset{H_1}{\gtrless}} T, \tag{8}$$

where

$$\begin{aligned} g(\|\boldsymbol{x}\|) &= \log\Bigg[\|\boldsymbol{x}\|^{\frac{c}{2}(\alpha_1+\alpha_2+N-1)-2N} \\ &\times \sum_{k=1}^{N} \frac{(-1)^{m+k}P_{(N,k)}}{\left(\frac{c}{2}(\sqrt{a}\|\boldsymbol{x}\|)^{\frac{c}{2}}\right)^{N-k}} \\ &\times K_{(\alpha_2-\alpha_1-k+1)}\left[2\left(\sqrt{a}\|\boldsymbol{x}\|\right)^{\frac{c}{2}}\right]\Bigg]. \end{aligned} \tag{9}$$

The function $g(\|\boldsymbol{x}\|)$ given above, is an inherent part of a system for detecting signals in GBK-distributed interference. The probability density functions $f[\Lambda(\boldsymbol{x})|\mathsf{H}_0]$ and $f[\Lambda(\boldsymbol{x})|\mathsf{H}_1]$ of the log-likelihood ratio test statistic $\Lambda(\boldsymbol{x})$ under the null hypothesis and the alternative cannot be derived in closed forms. Thus, in order to evaluate the performance of the detector, one has to use the empirical pdfs based on extensive computer simulations.

4. THE LLRT VERSUS UMPI BASED DETECTOR

As noted in the introduction, the detection of signals in GBK-distributed interference can be viewed as a detection of signals in conditionally Gaussian interference with some modulating random variate $S > 0$. Treating this modulating variable as an unknown parameter, the detection problem can be interpreted as one of detecting signals in Gaussian interference with unknown power. The UMPI decision rule in such a case, after a whitening transformation is applied, can be expressed as [7]

$$\Lambda(\boldsymbol{x}) = \frac{|\boldsymbol{x}\boldsymbol{p}^H|^2}{\boldsymbol{x}\boldsymbol{x}^H} \underset{H_0}{\overset{H_1}{\gtrless}} T. \tag{10}$$

It is known that the detector based on the rule given in (10) is equivalent to the one given in (8) for large N. In this work, the performance of both detectors was studied for small number of integrated pulses ranging from 2 to 16 and for different values of the parameters α_1, α_2, β, and c of the GBK distribution. As an example, Figures 1, 2, and 3, show the performance of the LLRT based detector (solid lines) and the UMPI detector (dashed lines) for different parameters of the GBK distribution and signal-to-noise ratios ranging from -15 dB to 20 dB.

Define the maximum improvement $\mathcal{I}_{\max}$ of the LLRT based detector over the UMPI detector as the maximum difference between the signal-to-noise ratios that result in the same probability of detection. Extensive simulations have shown that such defined improvement does not vary significantly when the parameters that control the property of spherical invariance (α_1 and c) of the GBK distribution are changed. However, the performance itself of the LLRT based detector depends on the parameters of the GBK distribution (except on the scale parameter β) and is mostly affected by changes in the power parameter c. In Table 1, the ceiling values of the maximum improvement are shown for different number of integrated pulses.

N	2	4	8	16
$\mathcal{I}_{\max}$ [dB]	18	8	4	3

Table 1: The maximum improvement of the LLRT based detector over the UMPI detector for different number of integrated pulses.

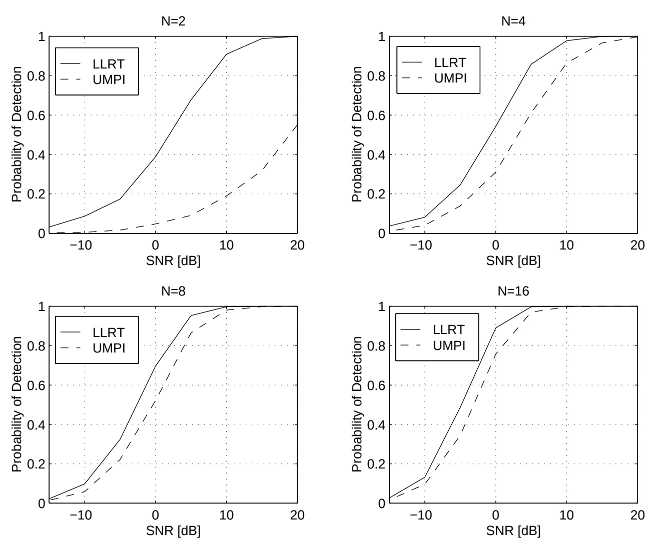

Figure 1: Performance of the LLRT based detector (solid lines) and the UMPI detector (dashed lines) for different number of integrated pulses, $P_{FA} = 10^{-3}$, and $\alpha_1 = 1$, $\alpha_2 = 2$, $\beta = 1$, and $c = 1$.

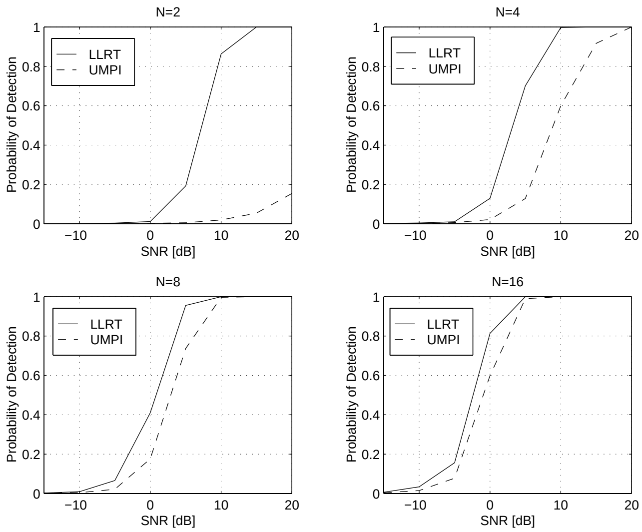

Figure 2: Performance of the LLRT based detector (solid lines) and the UMPI detector (dashed lines) for different number of integrated pulses, $P_{FA} = 10^{-3}$, and $\alpha_1 = 1$, $\alpha_2 = 2$, $\beta = 1$, and $c = 2$.

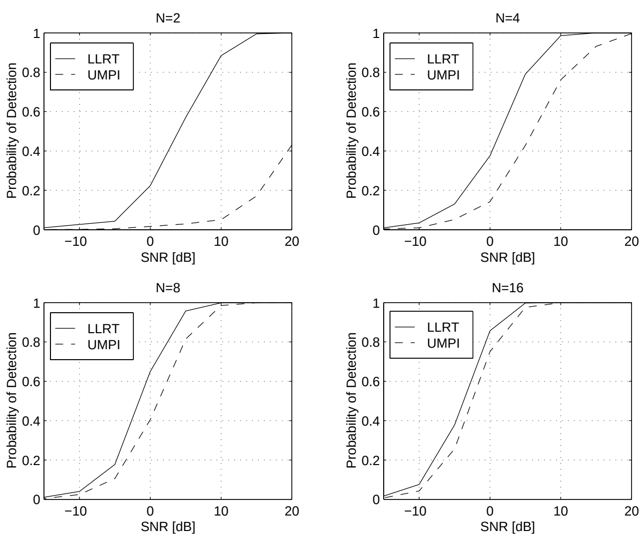

Figure 3: Performance of the LLRT based detector (solid lines) and the UMPI detector (dashed lines) for different number of integrated pulses, $P_{FA} = 10^{-3}$, and $\alpha_1 = 2$, $\alpha_2 = 2$, $\beta = 1$, and $c = 1$.

5. CONCLUSIONS

The detection of signals in GBK-distributed interference was considered. The detection scheme described is limited to the case where the GBK distribution fulfils the requirements of spherical invariance. However, this assumption does not diminish the applicability of the proposed detection schemes in practice as most physically derived interference models are in fact spherically invariant. An important feature of such a detector is fact that it has the same structure for all spherically invariant model included in GBK distribution.

The performance of the optimal detection scheme was compared to the uniformly most powerful invariant detector using extensive Monte Carlo simulations. It has been established that for small number of integrated pulses, say up to 8, the improvement of the LLRT over UMPI can reach up to 18 dB. This improvement has been found not to vary significantly with changes in the parameters of the GBK distribution. For $N \geq 16$ this improvement is negligible for most practical applications and it is not recommended there to use the amplitude information at all.

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