APPLICATION OF INFINITE DIMENSIONAL LINEAR PROGRAMMING TO FIR FILTER DESIGN WITH TIME DOMAIN CONSTRAINTS

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ABSTRACT

Previously the envelope-constrained filtering problem was formulated as designing an FIR filter such that the filter's L_2 norm is minimized subject to the constraint that its response to a specified input pulse lies within a prescribed envelope. In this paper, we recast this filter design problem as a frequency-domain L_∞ optimization problem with time-domain constraints. Motivations for solving this problem are given. Then recently developed infinite dimensional linear programming techniques are used for the design of the required FIR filter. For illustration, we apply the approach to a numerical example which deals with the design of an equalization filter for a digital transmission channel.

1. INTRODUCTION

In signal processing many filter design problems can often be cast as a constrained optimization problem where the constraints are defined by the specifications of the filter or the related output signal. These specifications can arise either from practical considerations or from the standards set by certain regulatory bodies (see, e.g., [4]). In this contribution, we are concerned with the envelope-constrained (EC) filtering problem. Our objective is to design an FIR filter $H(\omega)$ to process a given input signal $s(k)$ which is corrupted by additive random noise $n(k)$, see Fig. 1(a). The noiseless output $\psi(k)$ is required to fit into a prescribed pulse shape envelope defined by the lower and upper boundaries $\varepsilon^-(k)$ and $\varepsilon^+(k)$, see Fig. 1(b). Previously [6], the L_2 *optimal envelope-constrained filter* was defined as the filter which minimizes the output noise power while satisfying the pulse shape constraints. Assuming that the random noise is white with constant power spectrum density, it can be verified that the output noise power is proportional to the squared L_2 norm of the filter to be designed. Hence the L_2 optimal EC filtering problem can be posed as

$$\min \|H\|^2 \ \text{ subject to } \ \varepsilon_-(k) \leq \psi(k) \leq \varepsilon_+(k) \qquad (1)$$

where $k = 0, 1, \ldots, M \Leftrightarrow 1$.

In standards, the performance of a digital link is often specified in terms of a mask applied to the received signal,

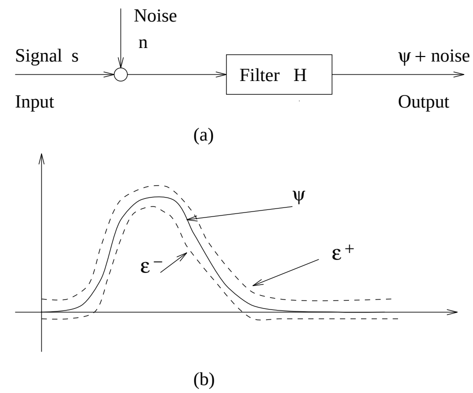

Figure 1: Envelope-constrained filtering problem: (a) Block diagram. (b) Pulse shape envelope

[2], [4], [7]. The envelope-constrained filter design problem is directly applicable and the input signal $s(k)$ would correspond to the test signal specified in the standard.

In this contribution, using the recently developed infinite dimensional linear programming techniques [1], [3], we shall design an FIR filter $H(\omega)$ such that its L_∞ norm, defined as $\|H\|_\infty = \max_{\omega\in[0,\pi]} |H(\omega)|$, is minimized subject to the same time-domain constraints as specified in (1).

The use of L_∞ norm arises naturally when the power spectrum of the exogenous input noise is bounded but otherwise unknown, while the use of L_2 norm in the EC filtering problem is relevant when the power spectrum of the exogenous input noise is known. It can be shown that the use of L_∞ norm offers the most robust design with respect to the worst case noise scenario, see e.g. [9]. Furthermore, the simplex algorithm for linear programming is generally more computationally efficient than the quadratic programming algorithms required for the L_2 EC design problem.

2. FILTER DESIGN FORMULATION

The frequency response of an N-tap FIR filter is given by

$$H(\omega) = \sum_{k=0}^{N-1} h(k) e^{-j\omega k} = \mathbf{h}^T \phi(\omega) \qquad \omega \in [0, \pi] \qquad (2)$$

where $\mathbf{h}$ is a real $N \times 1$ vector containing the impulse response $h(k)$, and $\phi(\omega)$ a complex vector of basis functions $e^{-j\omega k}$, $k = 0, \ldots, N-1$.

The discrete-time L_∞ optimal EC filter design problem can be formulated as follows:

Given an input signal $s(k)$, find an FIR filter $H(\omega)$ which solves the following constrained optimization problem

$$\min_{H} \|H\|_\infty^2 \text{ subject to } \varepsilon^-(k) \leq \psi(k) \leq \varepsilon^+(k) \qquad (3)$$

where $\psi(k) = \sum_{i=0}^{N-1} h(i)s(k-i),\ k = 0, 1, \ldots, M-1$. Note that compared to the L_2 formulation of the EC filtering problem (1), minimizing the squared L_∞ norm of the filter is equivalent to minimizing the output noise power corresponding to the worst case input noise n (see e.g. [9]). Therefore, in the situation where the power spectrum of the exogenous input noise is bounded but otherwise unknown, the L_∞ formulation of the EC filtering problem is directly applicable.

In order to obtain a nontrivial solution (i.e. $H(\omega) \neq 0$), we assume that there exists at least one k ($0 \leq k \leq M-1$) such that $\varepsilon^-(k)\varepsilon^+(k) > 0$. Furthermore, we assume that $\varepsilon^+(k) > \varepsilon^-(k),\ \ (k = 0, 1, \ldots)$.

We wish to put the L_∞ optimal EC filter design problem (3) in a more general setting which may also include a desired frequency response of the FIR filter designed. For this purpose we pose the following design criterion

$$\min_{\mathbf{h} \in R^N} \max_{\omega \in \Omega} v(\omega)|H_d(\omega) - H(\omega)| \qquad (4)$$

$$\text{subject to} \qquad \varepsilon^- \leq \mathbf{S}\mathbf{h} \leq \varepsilon^+ \qquad (5)$$

where Ω is a closed subset of the interval $[0, \pi]$, $v(\omega)$ a strictly positive weighting function, $H_d(\omega)$ the desired complex response, $\mathbf{S}$ a convolution matrix, and ε^- and ε^+ are vectors of lower and upper bounds $\varepsilon^-(k)$ and $\varepsilon^+(k)$, respectively. The convolution matrix $\mathbf{S}$ is defined so that $\psi = \mathbf{S}\mathbf{h}$, where ψ is the vector containing the noiseless output signal $\psi(k)$, $k = 0, 1, \ldots, M-1$. The L_∞ EC design problem (3) corresponds to the case with $\Omega = [0, \pi]$, $v(\omega) = 1$ and $H_d(\omega) = 0$.

According to the *real rotation theorem* [5], a magnitude inequality in the complex plane can be expressed in the following equivalent form.

$$|z| \leq \delta \quad \Leftrightarrow \quad \Re\left\{ze^{j\theta}\right\} \leq \delta \qquad \forall\theta \in [0, 2\pi] \qquad (6)$$

where z is a complex number, δ is a real and positive number, and $\Re\{\cdot\}$ denotes the real part.

By making use of (6), the nonlinear approximation problem (4)–(5) can be reformulated as the following continuous semi–infinite linear program

$$\left\{ \begin{array}{l} \min \delta \\ v(\omega)\Re\left\{(H_d(\omega) - H(\omega)) \cdot e^{j\theta}\right\} \leq \delta \\ \mathbf{P}\mathbf{h} \leq \mathbf{p} \end{array} \right. \qquad (7)$$

where δ is an additional real variable, $\omega \in \Omega$, $\theta \in [0, 2\pi]$, $\mathbf{P}^T = [\mathbf{S}^T\ -\mathbf{S}^T]$ and $\mathbf{p}^T = [\varepsilon^{+T}\ -\varepsilon^{-T}]$.

The linear program (7) is called (continous) semi–infinite since the constraint set is infinite (uncountable) and the number of variables is finite.

Define the $(N+1) \times 1$ vectors $\mathbf{y}$ (variable) and $\mathbf{b}$ (constant) by

$$\mathbf{y} = \begin{pmatrix} \mathbf{h} \\ \delta \end{pmatrix}, \qquad \mathbf{b} = \begin{pmatrix} \mathbf{0} \\ 1 \end{pmatrix} \qquad (8)$$

where $\mathbf{0}$ is a vector of zeros. Further, let the $N \times 1$ vector function $\mathbf{a}(\omega, \theta)$ and the scalar function $c(\omega, \theta)$ be defined by

$$\begin{aligned} \mathbf{a}(\omega, \theta) &= v(\omega)\Re\left\{\phi(\omega) \cdot e^{j\theta}\right\} & (9) \\ c(\omega, \theta) &= v(\omega)\Re\left\{H_d(\omega) \cdot e^{j\theta}\right\} & (10) \end{aligned}$$

The linear program (7) is now restated in the following form

$$\left\{ \begin{array}{l} \min \mathbf{b}^T\mathbf{y} \\ \begin{bmatrix} \mathbf{a}^T(\omega, \theta) & 1 \\ -\mathbf{P} & \mathbf{0} \end{bmatrix} \mathbf{y} \geq \begin{bmatrix} c(\omega, \theta) \\ -\mathbf{p} \end{bmatrix} \end{array} \right. \qquad (11)$$

where $\mathbf{0}$ is a $2M \times 1$ vector of zeros.

The filter design problem (4)–(5) can now be compactly formulated as the following (dual) continuous semi–infinite linear program

$$(\mathrm{D}) \ \left\{ \begin{array}{l} \min \mathbf{b}^T\mathbf{y} \\ \mathbf{A}_\alpha^T\mathbf{y} \geq c_\alpha \end{array} \right. \qquad (12)$$

where $\mathbf{A}_\alpha^T$ and c_α are the rows of the left hand side constraint matrix and the elements of the right hand side constraint vector in (11), respectively. Here α is an index belonging to a closed index set $\mathcal{A} \subset R^4$ which has a one–to–one correspondence with the constraint rows of (11). The set $\mathcal{A}$ can be constructed as follows. Let $\alpha = (\omega, \theta, i, l)$ where $l = 1$ or 2 since there are 2 types of constraints in (11). If $l = 1$, then $(\omega, \theta) \in \Omega \times [0, 2\pi]$ and $i = 1$. If $l = 2$, then $\omega = 0$, $\theta = 0$ and $i = 1, \ldots, 2M$. The formulation (12) is now in a form which has been thoroughly investigated [1].

We refer to (12) as the dual formulation since it corresponds to the dual of a linear program in standard form [1, 8].

3. THE SEMI–INFINITE SIMPLEX ALGORITHM

3.1. The Dual Formulations

The primal linear programming formulation corresponding to (12) is given by

$$(\mathrm{P}) \ \left\{ \begin{array}{l} \max \int c_\alpha dx_\alpha \\ \int \mathbf{A}_\alpha dx_\alpha = \mathbf{b} \\ x_\alpha \geq 0 \end{array} \right. \qquad (13)$$

where the maximization is with respect to the set of all regular Borel measures x_α [1]. The formulation (13) is said to be in standard form.

By introducing the semi–infinite matrix and vector notations $\mathbf{A} = (\mathbf{A}_\alpha)$, $\mathbf{c} = (c_\alpha)$, and $\mathbf{x} = (x_\alpha)$, and with the interpretation of inner products as in (13), the primal and dual problem (13) and (12) can now be formulated as

$$\text{(P)} \left\{ \begin{array}{l} \max \mathbf{c}^T\mathbf{x} \\ \mathbf{A}\mathbf{x} = \mathbf{b} \\ \mathbf{x} \geq 0 \end{array} \right. \qquad \text{(D)} \left\{ \begin{array}{l} \min \mathbf{b}^T\mathbf{y} \\ \mathbf{A}^T\mathbf{y} \geq \mathbf{c} \end{array} \right. \tag{14}$$

The notation for the dual pair in (14) is now in the same form as for finite–dimensional linear programs [8].

The duality theorem for finite–dimensional linear programs states that if $\mathbf{x}$ and $\mathbf{y}$ are feasible for (P) and (D), respectively, then $\mathbf{c}^T\mathbf{x} = \mathbf{b}^T\mathbf{y}$ iff $\mathbf{x}$ is optimal for (P) and $\mathbf{y}$ is optimal for (D), cf. [8]. This condition is also known as *strong duality*.

Strong duality does not necessarily hold for semi–infinite linear programs. Examples can easily be constructed for which there exists a *duality gap* between the optimal values of the dual programs in (14), cf. [1]. Fortunately, the semi–infinite linear program corresponding to the *uniform approximation problem* (7) satisfies strong duality. This is readily proven by applying theorem 4.4c in [1].

The success of the semi-infinite linear programming approach relies on the property of strong duality, but also on the equally important *Dimensionality Theorem* (Theorem 4.8 in [1]). This dimensionality theorem states that if there is a feasible solution to (P) in (13) with value z_0, then there is a feasible solution of finite support consisting of at most $N + 1$ points which achieves the same value.

The duality theorem enables us to solve the primal (P) rather than the dual (D) in (14). When the optimum solution to (P) is obtained we will also be in possession of the optimum solution to (D). The advantage of this procedure is that the primal (P) is in standard form for solution by the simplex algorithm. Furthermore, the dimensionality theorem enables the simplex algorithm to work with finite basic feasible solutions in very much the same way as in the case of finite–dimensional linear programming. The problem (P) is thus much easier to solve than (D).

3.2. The Revised Simplex Algorithm

The revised simplex algorithm works with four basic steps as described below. The procedure is identical with that for the finite–dimensional case, and is valid for the infinite–dimensional case provided that inner products of infinite–dimensional vectors are given the proper interpretation as defined above.

Given a *basis* $\mathbf{B} = [\mathbf{A}_{\alpha_1}, \ldots, \mathbf{A}_{\alpha_m}]$ where $\alpha_i \in \mathcal{A}$ for $i = 1, \ldots, m = N + 1$ and a basic feasible solution $\mathbf{x}_B \geq \mathbf{0}$ satisfying $\mathbf{B}\mathbf{x}_B = \mathbf{b}$. The dimension of $\mathbf{B}$ and $\mathbf{x}_B$ are $m \times m$ and $m \times 1$, respectively.

Let the matrix $\mathbf{A}$ be partitioned as $\mathbf{A} = [\mathbf{B}\ \mathbf{N}]$ where $\mathbf{N} = (\mathbf{A}_\alpha)$ consists of these columns of $\mathbf{A}$ which is not included in $\mathbf{B}$. Define the corresponding partition of the vectors $\mathbf{c}^T = [\mathbf{c}_B^T\ \mathbf{c}_N^T]$ and $\mathbf{x}^T = [\mathbf{x}_B^T\ \mathbf{x}_N^T]$. The primal cost is $\mathbf{c}^T\mathbf{x} = \mathbf{c}_B^T\mathbf{x}_B = \mathbf{c}_B^T\mathbf{B}^{-1}\mathbf{b}$.

The revised simplex algorithm proceeds as follows, cf. [8].

1. Calculate the dual variables by solving $\mathbf{B}^T\mathbf{y} = \mathbf{c}_B$, that is $\mathbf{y}^T = \mathbf{c}_B^T\mathbf{B}^{-1}$.
   The dual cost is $\mathbf{y}^T\mathbf{b} = \mathbf{c}_B^T\mathbf{B}^{-1}\mathbf{b} = \mathbf{c}^T\mathbf{x}$. If $\mathbf{y}$ is feasible for (D) then the current solution $\mathbf{x}_B$ is optimal for (P) according to the duality theorem. Stop.

2. Determine a column $\mathbf{A}_{\alpha'}$ to enter the basis by choosing an index α' so that $\mathbf{y}^T\mathbf{A}_{\alpha'} < c_{\alpha'}$ (constraint violation for (D)). The standard pivoting rule is to choose
$$\alpha' = \max_\alpha \left\{ c_\alpha \Leftrightarrow \mathbf{y}^T\mathbf{A}_\alpha \right\} \tag{15}$$

3. Determine a column $\mathbf{A}_{\alpha_j}$ to leave the basis by the ratio test
$$j = \arg \min_{i: d_i > 0} \left\{ \frac{x_{Bi}}{d_i} \right\} \tag{16}$$
   where the x_{Bi}s are the elements of the basic feasible solution $\mathbf{x}_B$ and the d_is are the coordinates of the entering column in terms of the old basis. Thus, $\mathbf{B}\mathbf{d} = \mathbf{A}_{\alpha'}$ where $\mathbf{d} = (d_i)$.

4. Update the basis matrix $\mathbf{B}$ by replacing the column $\mathbf{A}_{\alpha_j}$ for $\mathbf{A}_{\alpha'}$. $\square$

The representation of the vector $\mathbf{b}$ in the new basis is given by
$$\mathbf{B}(\mathbf{x}_B \Leftrightarrow t \cdot \mathbf{d}) + t \cdot \mathbf{A}_{\alpha'} = \mathbf{b} \tag{17}$$
where $t \geq 0$ is the minimum ratio in (16) in step 3 above. The ratio test ensures that the new basic solution is feasible, that is, $\mathbf{x}_B \Leftrightarrow t \cdot \mathbf{d} \geq \mathbf{0}$ with equality for at least one component (j). The *degenerate* case is when $t = 0$.

Convergence of the algorithm above is guaranteed [1, 8], except for the possibility of cycling of degenerate basic solutions (in which case $t = 0$). However, cycling is a rarely occurring phenomenon, and there are several well established remedies, see e.g. Bland's rule [8].

Suppose that there are no additional linear constraints (5) to the problem (4). The column vector $\mathbf{A}_\alpha$ is then given by $\mathbf{A}_\alpha^T = [\mathbf{a}^T(\omega, \theta)\ 1]$ and the scalar $c_\alpha = c(\omega, \theta)$. Using (15) in step 2 above and the definitions given in (9) and (10), the explicit relations for the variables (ω', θ') corresponding to the column entering the basis are given by

$$\begin{aligned} \omega' &= \arg\max_{\omega \in \Omega} \left\{ v(\omega) |H_d(\omega) \Leftrightarrow H(\omega)| \right\} && (18) \\ \theta' &= \Leftrightarrow \arg \left\{ H_d(\omega') \Leftrightarrow H(\omega') \right\} && (19) \end{aligned}$$

The relations (18) and (19) are easily extended to include additional linear constraints as in (11).

The revised simplex algorithm as outlined above assumes that an initial basic feasible solution to (P) is available. This is phase two of the two–phase simplex algorithm. In phase one, a basic feasible solution to (P) is found by solving the so called artificial minimization problem [8].

4. A DESIGN EXAMPLE

As a numerical example for the proposed L_∞ EC FIR filter design procedure we consider the design of an equalization filter for a digital transmission channel consisting of a coaxial cable on which data is transmitted according to the DSX–3 standard [2, 4]. The design objective is to find an equalizing filter which takes a sampled impulse response of a coaxial cable with a loss of 30 dB at a normalized frequency of $1/T$ as input and produces an output which lies within the envelope given by the DSX–3 pulse template [2, 4].

Fig. 2 shows the coaxial cable impulse response s, the output envelope ε and the resulting noiseless output signal ψ. The solution corresponds to an L_∞–design with $N = 20$ coefficients obtained with the semi–infinite simplex procedure. Fig. 3 shows the resulting frequency response of the equalizer designed. The frequency response of the corresponding L_2–design is included for comparison.

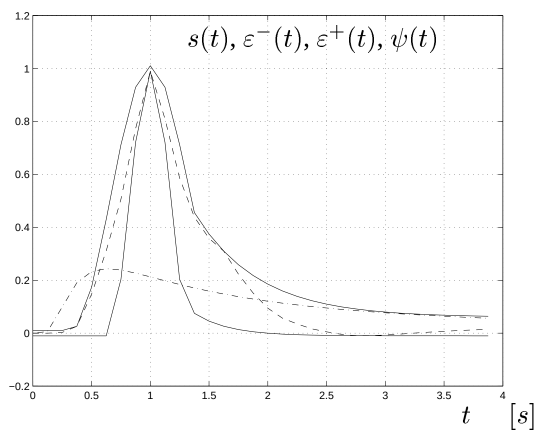

Figure 2: Input signal s (dash–dotted line), specified output envelope ε (solid line) and resulting noiseless output signal ψ (dashed line).

5. SUMMARY

The envelope–constrained (EC) filter design problem has been formulated as a special case of a general frequency domain L_∞ optimization problem. The optimization problem is cast as a semi–infinite linear program which can be solved by using numerically efficient simplex extension algorithms. A numerical example is included to demonstrate the efficiency of the design method.

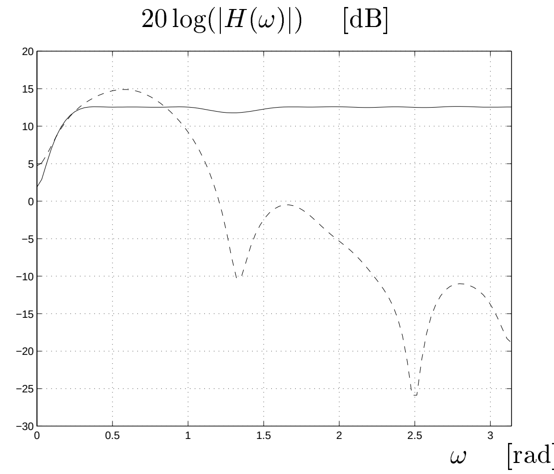

Figure 3: Resulting frequency response of equalizer for an L_∞–design (solid line), and an L_2–design (dashed line).

6. REFERENCES

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