# **BOOTSTRAP MODEL SELECTION FOR POLYNOMIAL PHASE SIGNALS**

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### ABSTRACT

We consider the problem of estimating the order of the phase of a complex valued signal, having a constant amplitude and a polynomial phase, measured in additive noise. A new method based on the bootstrap is introduced. The proposed approach does not require knowledge of the distribution of the noise, is easy to implement, and unlike existing techniques, it achieves high performance when only a small amount of data is available. The proposed technique can be easily extended to non-stationary signals which have a polynomial amplitude and phase, provided a consistent estimator for the parameters can be obtained.

# 1. INTRODUCTION

Non-stationary signals encountered in communications, sonar, radar, speech and biology often involve amplitude and/or the frequency modulation of a carrier signal. The so-called amplitude modulated-frequency modulated signal can be written as the complex representation  $z(t) = a(t) \exp\{j\phi(t)\}$ , where a(t) and  $\phi(t)$ ,  $t \in [T_1, T_2]$ ,  $T_1, T_2 < \infty$ , are the amplitude and phase functions. In practice, the signal z(t)is embedded in stationary complex noise U(t) so that the received signal is given by

$$X(t) = z(t) + U(t), \quad T_1 \le t \le T_2.$$
 (1)

Assume that the amplitude of z(t) is constant<sup>1</sup> and that the signal is appropriately sampled, yielding values  $x_t$ ,  $t = 0, \ldots, n-1$ , which are observations from

$$X_t = a_0 \exp\{j \sum_{k=0}^q b_k t^k\} + U_t, \quad t = 0, \dots, n-1, \quad (2)$$

where  $a_0$  is real valued, representing the amplitude,  $b_k$ ,  $k = 0, \ldots, q$ , are the real valued phase coefficients of the signal and q is a fixed integer. The objective is to determine the order of non-linearity of the phase based on estimates of the unknowns  $a_0, b_0, \ldots, b_q$ , given observations  $x_t$ ,  $t = 0, \ldots, n-1$ .

A classical approach for estimating the parameters of the model in (2) is the method of maximum likelihood. This, however, requires the distribution of the complex noise  $U_t$   $t = 0, \pm 1, \pm 2, \ldots$ , to be known. Often  $U_t$  is assumed to be white and Gaussian. Also, maximum likelihood algorithms are very complicated for a phase order  $q \ge 3$ . These difficulties have led to the development of parameter estimation techniques that are suboptimal, but are computationally inexpensive. Suboptimal methods for estimating the parameters of a constant amplitude polynomial phase signal were reported, for example, in [1, 4]. Among these techniques, the only one that addresses the problem of order selection is the one based on the polynomial phase transform, proposed by Peleg and Porat [5]. It gives bases for estimating the order q of the polynomial phase. However, this model selection method relies on the Cramér-Rao bound and thus assumes the distribution of the noise to be known.

Many model order selection procedures exist in the case of a linear model, which may be obtained for the phase of (2) at high signal-to-noise power ratio. Popular techniques are Akaike's information criterion, Rissanen's minimum description length criterion, and Hannan and Quinn's criterion (see for example [3]). These criteria are often used in the context of estimating the parameters p and q of an autoregressive moving average process of order (p, q). Experimental as well as theoretical results indicated that the model criteria do not yield definitive results. In the absence of any prior information regarding the physical process that resulted in the data, one is often left with trying different model orders and different criteria and, ultimately, interpreting the different results.

In this paper we introduce an approach for selecting the order of a constant amplitude, polynomial phase signal using the bootstrap [2, 9]. Besides the theoretical and empirical properties of bootstrap selection procedures such as the ones discussed in [7], there are good reasons to use a bootstrap model selection procedure. Bootstrap methods are simple and computationally efficient (see for example [11]). If one uses a bootstrap approach for the model selection and for the subsequent inference, then the bootstrap observations generated for model selection can also be used in the inference procedure. Thus, the model selection procedure can be done at no extra computational cost.

Our approach for model selection is powerful in situations where the sample size is small and the distribution of the noise is unknown. The only assumption we make is that the noise sequence is identically and independently distributed. However, we will discuss later how this assump-

<sup>&</sup>lt;sup>1</sup>The more complex case where a(t) is time-varying, modeled as a polynomial of finite order will be briefly discussed in section 4.

tion can be alleviated. The results presented in this paper are based on a constant amplitude, polynomial phase signal, but can easily be extended to non-stationary signals which have a polynomial amplitude and phase, provided a consistent estimation technique for the parameters of such signals is available. An outline of the paper follows.

Section 2 introduces our approach to selecting the model order of a constant amplitude, polynomial phase signal, using the bootstrap. In Section 3, we present results with simulated data and measure the performance with the empirical probabilities of selecting various models. We briefly discuss how the techniques presented in section 4 can be extended to model non-stationary signals, having a polynomial amplitude and a polynomial phase.

## 2. A BOOTSTRAP APPROACH FOR MODEL SELECTION

## 2.1. Parameter Estimation

For the sake of clarity, we will assume that the signal-topower ratio (SNR), defined to be  $a_0^2/\sigma_U^2$ , where  $\sigma_U^2$  is the power of the noise, is sufficiently high. This assumption can be alleviated, but the bootstrap procedure for model order selection to be introduced in the next subsection will need to be altered to cater for other estimators (see section 4). Also, and without loss of generality, we assume that  $a_0 = 1$ .

Let the phase of  $z_t$ , described in (2), be given by

$$\phi_t = \sum_{k=0}^q b_k t^k, \qquad t = 0, \dots, n-1,$$
 (3)

 $\mathbf{b} = (b_0, \dots, b_q)'$  be a vector of q + 1 phase coefficients and q be a fixed integer. Under the assumption that the SNR is large one can approximate the signal given in (2) by

$$X_t \approx \exp\{j(\phi_t + W_t)\} = \exp\{j\hat{\phi}_t\},\qquad(4)$$

with  $W_t$  being a real zero-mean independent noise sequence with variance  $\sigma_W^2 = \sigma_U^2/2$  [8, 1].

In the sequel we will base our procedure on parameter estimates obtained through the method of least squares, which requires that the phase be unwrapped [1]. The unwrapped phase  $\hat{\phi}_t$ ,  $t = 0, \dots, n-1$ , can be written in the matrix form

$$\hat{\Phi} = \mathbf{H} \mathbf{b} + \mathbf{W},\tag{5}$$

where  $\hat{\Phi} = (\hat{\phi}_0, \dots, \hat{\phi}_{n-1})', \mathbf{W} = (W_0, \dots, W_{n-1})'$  and

$$\mathbf{H} = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 1 & 1 & 1 & \cdots & 1 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 1 & n-1 & (n-1)^2 & \cdots & (n-1)^q \end{pmatrix}.$$

The least-squares estimate for b is then given by

$$\hat{\mathbf{b}} = (\mathbf{H}'\mathbf{H})^{-1}\mathbf{H}'\hat{\Phi}.$$
 (6)

#### 2.2. Model Selection

In practical situations the model order q is unknown. We wish to select a subset of the parameters  $\{b_k\}, k = 0, \ldots, q$ , to fit  $\hat{\phi}_t$  to the model given in (3). The problem is stated as follows: given  $\hat{\phi}_0, \ldots, \hat{\phi}_{n-1}$ , estimate the parameter q. This can be formulated as a model selection problem in which we select  $\beta$  from  $\{0, \ldots, q\}$  and each  $\beta$  corresponds to the model in (4) of order  $\beta$ , i.e.,

$$\hat{\phi}_t = \sum_{k=0}^q b_k t^k + W_t, \qquad t = 0, \dots, n-1.$$

Under  $\beta$ , we have  $\mathbf{b}_{\beta} = (b_0, \dots, b_{\beta})'$  which is obtained by the least squares estimate

$$\hat{\mathbf{b}}_{\beta} = (\mathbf{H}_{\beta}'\mathbf{H}_{\beta})^{-1}\mathbf{H}_{\beta}'\hat{\Phi}.$$
(7)

where the expression for  $\mathbf{H}_{\beta}$  is similar to that of  $\mathbf{H}$ , except that q is replaced by  $\beta$  in  $\mathbf{H}$ . We assume that  $\beta = q$  is the largest possible model. The *optimal* model is

$$\beta_0 = \max\{k : 0 \le k \le q, \ b_k \ne 0\}.$$

Let  $w_t^*$ ,  $t = 0, \pm 1, \pm 2, \ldots$  be i.i.d. from the distribution putting mass  $n^{-1}$  to  $\hat{r}_t - 1/n \sum_{t=0}^{n-1} \hat{r}_t$ , where  $\hat{r}_t = \hat{\phi}_t - \sum_{k=0}^{q} \hat{b}_k t^k$  is the *t*:th residual under the largest model  $\beta = q$ . The bootstrap analog  $\hat{\mathbf{b}}_{\beta,m}^*$  of  $\hat{\mathbf{b}}_{\beta}$  is defined in (7) with *n* replaced by *m* and  $\hat{\Phi}$  replaced by  $\hat{\Phi}^*$ , whose elements are given by

$$\hat{\phi}_t^* = \sum_{k=0}^{\beta} \hat{b}_k t^k + \hat{w}_t^*, \qquad t = 0, \dots, m-1.$$
 (8)

The model selected by the bootstrap is the minimizer of

$$\bar{\Gamma}_{n,m}(\beta) = \mathsf{E}_* \sum_{t=0}^{n-1} \frac{\left(\hat{\phi}_t - \sum_{k=0}^{\beta} \hat{b}_{k,m}^* t^k\right)^2}{n} \qquad (9)$$

over  $\beta = 0, ..., q$ , where  $\mathsf{E}_*$  denotes expectation with respect to the bootstrap sampling [7]. This procedure is consistent in the sense that

$$\lim_{n \to \infty} \mathsf{P}\{\hat{\beta} = \beta_0\} = 1,$$

provided m satisfies  $m/n \rightarrow 0$  and  $m \rightarrow \infty$ . For practical uses, m needs to be specified for a fixed n. One should choose m so that the least squares fitting of model (6) does not have too high variability [7]. One possible choice of m could be  $m = n^{\gamma}$ ,  $0 < \gamma < 1$ . For more details on the choice of m the reader is referred to [7]. Computation of  $\overline{\Gamma}_{n,m}(\beta)$  in (9) can be performed through Monte Carlo approximation. Given B bootstrap resamples  $\hat{\phi}_t^*$  for  $t = 0, \ldots, n - 1, \overline{\Gamma}_{n,m}(\beta)$  can be approximated by

$$\bar{\Gamma}_{n,m}^{(B)}(\beta) = \frac{1}{B} \sum_{b=1}^{B} \frac{\|\hat{\Phi} - \mathbf{H}_{\beta}\hat{\mathbf{b}}_{\beta,m}^{*b}\|^2}{n}$$
(10)

for all  $\beta \in \{0, \ldots, q\}$ . A detailed procedure for the model selection is given in Table 1. Notice that the method of resampling used in Table 1 is based on the i.i.d. assumption of the noise sequence. In the case where this assumption does not hold, we could use the method of *subsampling*, suggested in [6], which works for a colored noise sequence under minimal assumptions.

The bootstrap data generated for computing (10) can be used in inference, for example, for setting confidence intervals for b after model selection. A method for setting confidence intervals for the parameters of a constant amplitude, polynomial phase signal under the assumption of a known model order has been introduced in [10].

# 3. SIMULATION RESULTS.

The procedure described in Table 1 was tested extensively on polynomial phase signals whose amplitude is constant. We report here results obtained for a signal that has a cubic (q = 3) phase, and a constant amplitude, in the presence of Gaussian and double exponential noise. We consider data lengths of n = 64 and n = 32 and a varying SNR within the range 5-15 dB. For each SNR, B = 100 bootstrap resamples of the signal were produced and the procedure was repeated 100 times to find the probability of correct phase order selection. When estimating the residuals, q was set to 6 and m was chosen to be 48 and 24 for n = 64 and 32, respectively.

The results are shown in Tables 2 and 3 for Gaussian and double exponential noise, respectively, when n = 64. They indicate that the empirical probability of correctly selecting the phase order is high at a reasonable SNR. As expected, the noise distribution does not seem to have much influence on the performance. For n = 32, Table 4 shows that the results are satisfactory for an SNR of 15 and 10 dB. The results for 5 dB SNR are not as good as in the case when n = 64, which is not surprising for such a small amount of data.

It was not expected that the method would perform well at low SNR (0 dB) because it is based on the least squares estimator of the phase, and a least squares estimator requires an SNR larger than approximately 5 dB for the approximation in (4) to be accurate [1, 8]. An alternative bootstrap model selection approach in the case of a low SNR is discussed in section 4.

The choice of the largest possible model had little effect on the results as long as this number is larger than the true model order. Also, similar results were obtained for other values for m, smaller than n. The results obtained so far confirm that bootstrap techniques can be employed to model selection of signals, having a constant amplitude and a polynomial phase, when the distribution of the noise sequence is unknown and the sample size is small. An extension of our approach to the case where approximation (4) is inaccurate, or to signals whose amplitude and phase can be modeled by polynomials is discussed next.

Table 1: Bootstrap based procedure for estimating the order of a polynomial phase signal.

**Step 1.** Select the largest possible model  $\beta = q$ , and find the least squares estimate  $\hat{\mathbf{b}}$  of  $\mathbf{b} = (b_0, \dots, b_q)'$ .

Step 2. Compute the residuals

$$\hat{r}_t = \hat{\phi}_t - \sum_{k=0}^q \hat{b}_k t^k$$
,  $t = 0, \dots, n-1$ .

Step 3. Center the residuals, to obtain

$$\tilde{r}_t = \hat{r}_t - \frac{1}{n} \sum_{t=0}^{n-1} \hat{r}_t, \quad t = 0, \dots, n-1.$$

**Step 4.** For all  $0 \le \beta \le q$ ,

- (a) draw independent bootstrap residuals  $\tilde{r}_t^*$ , with replacement, from the empirical distribution of  $\tilde{r}_t$ , t = 0, ..., n 1.
- (b) Compute

$$\hat{\phi}_t^* = \sum_{k=0}^{\beta} \hat{b}_k t^k + \tilde{r}_t^*, \quad t = 0, \dots, m-1,$$

where m is such that  $m/n \to 0$  and  $m \to \infty$ .

- (c) Find the least squares estimate  $\hat{\mathbf{b}}_{\beta,m}^*$  from (b).
- (d) Compute

$$\hat{\Gamma}_{n,m}^{*}(\beta) = \frac{1}{n} \sum_{t=0}^{n-1} \left( \hat{\phi}_t - \sum_{k=0}^{\beta} \hat{b}_{k,m}^{*} t^k \right)^2.$$

(e) Repeat steps (a)–(d) a large number of times (e.g. 100), to obtain a total of *B* bootstrap statistics

$$\hat{\Gamma}_{n,m}^{*(1)}(\beta),\ldots,\hat{\Gamma}_{n,m}^{*(B)}(\beta)$$

and

$$\bar{\Gamma}_{n,m}(\beta) = \frac{1}{B} \sum_{b=1}^{B} \hat{\Gamma}_{n,m}^{*(b)}(\beta) \,.$$

**Step 5.** Choose  $\beta$  for which  $\overline{\Gamma}_{n,m}(\beta)$  is minimum with respect to  $\beta$ .

Table 2: Estimates of the probability (in percent) of selection of order for a cubic phase signal (q = 3) in Gaussian noise, when n = 64.

SNR [dB] / $\beta$	0	1	2	3	4	5	6
15	0	0	0	100	0	0	0
10	0	0	0	100	0	0	0
5	0	3	0	97	0	0	0
0	0	76	18	6	0	0	0

Table 3: Estimates of the probability (in percent) of selection of order for a cubic phase signal (q = 3) in double exponential noise, when n = 64.

SNR [dB] / $\beta$	0	1	2	3	4	5	6
15	0	0	0	100	0	0	0
10	0	0	0	100	0	0	0
5	0	8	0	92	0	0	0
0	0	61	11	28	0	0	0

# 4. EXTENSIONS OF THE BOOTSTRAP MODEL SELECTION

In the case where the SNR is not sufficiently high and thus, approximation (4) is not valid, we will alter the procedure given in Table 1 in the following manner. Using an appropriate estimation technique for the phase and amplitude parameters, which leads to consistent estimators, we first estimate  $a_0, b_0, \ldots, b_q$ , using the largest possible order  $\beta = q$ , to yield  $\hat{a}_0, \hat{b}_0, \ldots, \hat{b}_q$ . With these estimates, we compute  $\hat{z}_{t,q} = \hat{a}_0 \exp\{j \sum_{k=0}^{q} b_k t^k\}$ , and the residuals  $\hat{\varepsilon}_t = x_t - \hat{z}_{t,q}, t = 0, \ldots, n-1$ . These are centered, to obtain  $\tilde{\varepsilon}_t$ . For all  $0 \le \beta \le q$ , we then draw resamples  $\tilde{\varepsilon}_t^*$ , with replacement, from  $\tilde{\varepsilon}_t$ , compute  $x_t^* = \hat{z}_{t,\beta} + \tilde{\varepsilon}_t^*, t = 0, \ldots, m-1$ , and estimate  $\hat{a}_{0,m}^*, \hat{b}_{0,m}^*, \ldots, \hat{b}_{\beta,m}^*$ , which lead to  $\hat{z}_{t,\beta}^*$ . The optimal order is then the minimizer over  $\beta = 0, \ldots, q$  of

$$\bar{\Gamma}_{n,m}(\beta) = \mathsf{E}_* \sum_{t=0}^{n-1} \frac{\left(x_t - \hat{z}_{t,\beta}^*\right)^2}{n}.$$

We will proceed similarly if the amplitude can be modeled by a polynomial of finite order. Then, the search for a minimum will be with respect to, say,  $\alpha$  and  $\beta$ , which represent the variations of the amplitude and the phase order, respectively. However, for the method to be successful, one would require consistent estimators for the amplitude and phase parameters  $a_0, \ldots, a_{\alpha}$  and  $b_0, \ldots, b_{\beta}$ .

# 5. CONCLUSIONS

A method based on the bootstrap has been proposed for estimating the optimal order of the phase of a constant amplitude, polynomial phase signal. Unlike existing techniques, Table 4: Estimates of the probability (in percent) of selection of order for a cubic phase signal (q = 3) in double exponential noise, when n = 32.

SNR [dB] / $\beta$	0	1	2	3	4	5	6
15	0	0	0	100	0	0	0
10	0	3	0	97	0	0	0
5	0	44	10	46	0	0	0
0	1	80	14	5	0	0	0

the proposed method does not require the distribution of the interfering noise to be known. The results show that the bootstrap is able to estimate the true order with high probability for small sample sizes at moderate signal-to-noise power ratio. The introduced method was developed based on a least squares estimator of the phase which requires the phase to be unwrapped. This is no limitation and can be easily extended to other estimators. We also suggested an extension of the methodology to non-stationary signals which have polynomial amplitude and polynomial phase.

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