Asymptotically Perfect Reconstruction in Hybrid Filter Banks

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ABSTRACT

A procedure to derive a hybrid filter bank from a digital filter bank is presented. Perfect reconstruction is shown to be possible only asymptotically. The stability of the analog filters is ensured if FIR or IIR stable filters are used in the digital prototype.

I. INTRODUCTION

The generalized sampling theorem of Papoulis [1], states that a deterministic continuous-time band-limited signal $x(t)$ is uniquely determined from the outputs of M linear systems, sampled at $1/M$ the Nyquist rate. The concept of multi-channel sampling of Brown, formulates the same property in terms of M analysis and M synthesis analog filters constituting an analog filter bank [2]. In this formulation, the signal band is divided into M equal subbands. Then, the transfer function of the synthesis filters in each subband is obtained as a function of the analysis filters by solving a linear system of equations. The generalized sampling theorem has been also extended to two-dimensional signals [3] and linear T-periodic time-varying systems [4]. The generalized sampling theorem has been believed ill-posed because if one of the synthesis filters has infinite energy and the input signal is corrupted by additive noise, the output noise energy will not be bounded [5]. Nevertheless, it has been shown that if all of the synthesis filters have finite energy, the generalized theorem is well-posed, e.g.; the output noise energy is bounded [6]. This property has been further used to optimize the analysis filters for the reconstruction of oversampled noisy signals [7]. The generalized sampling theorem has been also demonstrated using a digital approach [8] and a relation between digital and analog filter banks has been established [9]. Digital filter banks have been studied extensively [10] and several design techniques with different objectives have been proposed.

Here, we study the signal reconstruction problem in hybrid filter banks composed of analog and digital filters. Because of flexibility and simplicity of design of digital filter banks, we will study hybrid filter banks by referring them to the digital filter of Fig.1. In all of the following sections, the quantization noise is neglected.

II. ANALOG TOWARDS DIGITAL HYBRID FILTER BANK

In the hybrid filter bank of Fig.2, the analog analysis filters are continuous-time linear systems represented by their transfer functions $H_m(s)$, $s = j\Omega$. The digital synthesis filters are defined by: $F_m(z)$, $z = exp(j\omega)$. For the sake of simplicity, suppose that the analog input signal is bandlimited to $|\Omega| \le \pi$. So, the Nyquist angular frequency is $\Omega_N = 2\pi$. Now, consider the discrete-time signal $x[n]$ obtained by sampling the analog signal $x(t)$ with $t = nT$, $T = 1$, Fig.1. The analog and discrete frequencies are related to each other by: $\omega = \Omega T = \Omega$. The output signal of the analysis filters in the digital filter bank may be written using the discrete-time inverse Fourier transform as:

$$x_m[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X\left(e^{j\omega}\right)H_m\left(e^{j\omega}\right)e^{j\omega}d\omega \qquad (1)$$

Noting the bandlimited supposition for the analog signal, the output of the analysis filters in the hybrid filter bank is obtained using continuous-time Fourier transform:

$$x_m(t) = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(j\Omega)H_m(j\Omega)e^{j\Omega t}d\Omega \qquad (2)$$

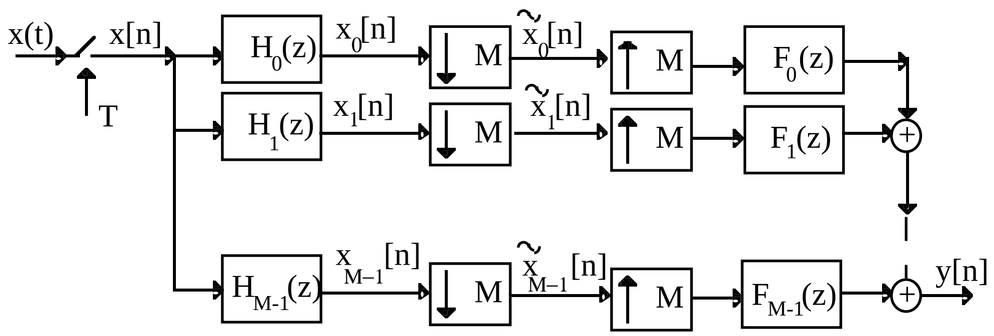

Fig.1, Digital filter bank for A/D conversion.

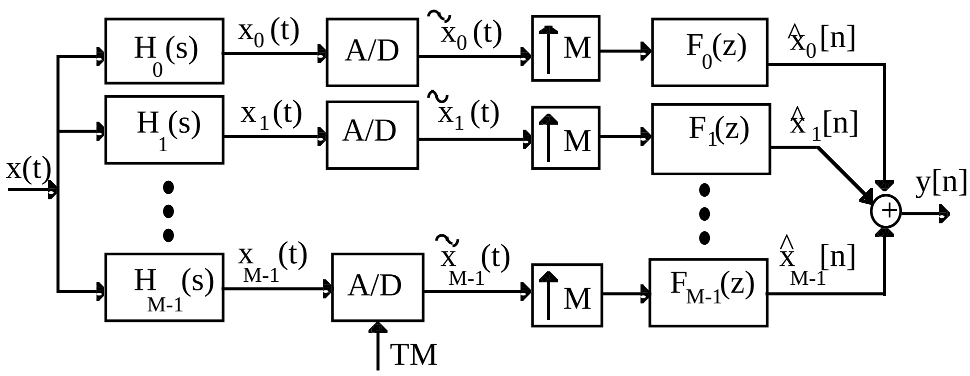

Fig.2, Hybrid filter bank for A/D conversion.

If we put

$$H_m(j\Omega) = H_m\left(e^{j\omega}\right)\Big|_{\Omega=\omega} \quad : \ |\Omega| \le \pi \tag{3}$$

for $t = nT$, $T = 1$, we have $x_m(nT) = x_m[n]$, and after sampling in the hybrid system or down-sampling in the digital system, respectively: $x_m(MnT) = x_m[Mn]$ or $\tilde{x}_m(nT) = \tilde{x}_m[n]$. Therefore with the condition (3), each analysis filter of Fig.2 can be replaced by its equivalent in Fig.1 without modifying the input samples of the expanders. It should be noted that the condition (3) must be valid only in the band $|\Omega| \le \pi$ and the behavior of the function $H_m(j\Omega)$ for $|\Omega| > \pi$ has no importance. This is due to the bandlimited signal supposition. The resulting hybrid system will conserve the properties of the digital prototype. If the digital filter bank is a perfect reconstruction system, the hybrid filter bank will be also a perfect reconstruction system and so on. It is clear that the relation (3) can be satisfied only approximately. Higher approximation orders offer better signal reconstruction. Hence, the system tends asymptotically towards perfect reconstruction. In summary, the hybrid filter bank can be designed by choosing a suitable digital system and then replacing the digital analysis filters by continuous-time analog filters using the relation (3). Now, we calculate the spectrum of the output signal. The output of each analysis filter is given by $X_m(j\Omega) = H_m(j\Omega)X(j\Omega)$. The output of the analysis filters is then sampled at the rate $1/TM$, so the sampled signals at the input of the expanders are given by:

$$\tilde{X}_m\left(e^{j\omega}\right) = \frac{1}{M}\sum_{n=-\infty}^{\infty} X\left(j\frac{\omega}{M} - j\frac{2\pi n}{M}\right) H_m\left(j\frac{\omega}{M} - j\frac{2\pi n}{M}\right) \tag{4}$$

The output signal is deduced by summing the signals $\hat{X}_m(e^{j\omega}) = X_m(e^{jM\omega})F_m(e^{j\omega})$. Note that due to the supposition of bandlimited signal, it is possible to carry out the summation in (2) only for $n = -(M-1), \ldots, (M-1)$ [7, 9]. Hence, we obtain:

$$Y\left(e^{j\omega}\right) = \underbrace{\frac{1}{M} X(j\omega)\sum_{m=0}^{M-1} H_m(j\omega)F_m\left(e^{j\omega}\right)}_{Distortion} +$$
$$\underbrace{\frac{1}{M}\sum_{\substack{n=-(M-1)\\ n\neq 0}}^{M-1} X\left(j\omega - j\frac{2\pi n}{M}\right)\sum_{m=0}^{M-1} H_m\left(j\omega - j\frac{2\pi n}{M}\right)F_m\left(e^{j\omega}\right)}_{aliasing} \tag{5}$$

III. DIGITAL TOWARDS ANALOG HYBRID FILTER BANK

The hybrid filter bank concept can be applied as well to D/A conversion. A hybrid filter bank for D/A conversion is represented in Fig.3. The output of the analog synthesis filters is given by the convolution of the continuous-time signals $\tilde{x}_m(t)$ with the continuous-time inverse Fourier transform of the synthesis filters transfer function $f_m(t)$: $\hat{x}_m(t) = f_m(t) \otimes \tilde{x}_m(t)$. The signals $\tilde{x}_m(t)$ are obtained by converting the samples $\tilde{x}_m[n]$ into impulses with a gain M: $\tilde{x}_m(t) = \sum_{l=-\infty}^{\infty} \tilde{x}_m[l]\delta(t - lM)$. Combining these relations and noting that the output signal is obtained by summing the signals $\hat{x}_m(t)$, one obtain:

$$y(t) = \sum_{m=0}^{M-1}\sum_{l=-\infty}^{\infty} \tilde{x}_m[l] f_m(t - lM) \tag{6}$$

From the theory of multirate systems [10], the output of the digital system of Fig.1 is written as:

$$y[n] = \sum_{m=0}^{M-1}\sum_{l=-\infty}^{\infty} \tilde{x}_m[l] f_m[n - lM] \tag{7}$$

The two above expressions are equivalent for $t = nT = n$ if $f_m[n] = f_m(nT)$. This means that the continuous signal $y(t)$ is obtained from the discrete signal $y[n]$ by an ideal D/A conversion with $T = 1$. $f_m[n]$ is given by inverse discrete Fourier transform:

$$f_m[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} F_m\left(e^{j\omega}\right) d\omega \tag{8}$$

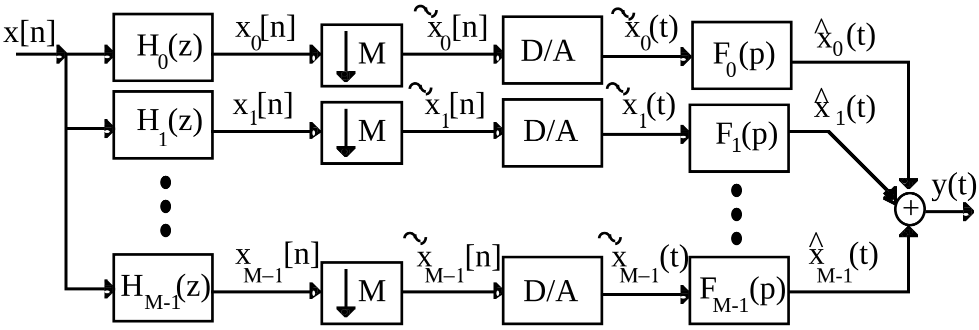

Fig.3, Hybrid filter bank for digital-to-analog conversion.

In the frequency domain, the equivalency condition is written as:

$$F_m(j\Omega) = \begin{cases} F_m(e^{j\omega}) & : |\Omega| \le \pi \\ 0 & : |\Omega| > \pi \end{cases} \tag{6}$$

It is not surprising to obtain continuous-time transfer functions needing to be zero beyond the signal band. Because, in reality, D/A conversion is composed of two operations: conversion of discrete samples into impulses and then ideal low-pass filtering.

IV. STABILITY OF ANALOG FILTERS

In the both cases of analog and digital filter banks, the synthesis filters allowing the perfect reconstruction are uniquely determined from the analysis filters. In analog filter banks, the synthesis filters are obtained by dividing the signal band into M subbands. Besides, the stability of the filters is not ensured by the generalized sampling theorem. In hybrid filter banks, due to relations (3) and (6) we don't need to divide the signal band into subbands in order to obtain the analog filters. In addition, the stability of analog filters is ensured if the prototype digital filter bank is composed of FIR or stable IIR filters. To prove this, we use the properties of the Padé approximation [11]. Let us represent the Padé approximants of an exponential function e^{-s} by $R_{P,Q}(s)$, where P and Q are the degree of the numerator and denominator, respectively. In the case of FIR filters, the rational functions in s must approximate transfer functions of the form $\sum a_n z^{-n} = \sum a_n e^{-ns}$. It has been already shown that the approximant $R_{P,Q}(s)$ has all its poles in the left half plane if $P \le Q+4$ [12]. In the case of IIR filters, we mention the property that $|z| = |R_{P,Q}(s)|^{-1} < 1$ for $Q = P$, $Q = P+1$ and $Q = P+2$ if $Re(s) < 0$ [13, 14]. This means that if the poles of a transfer function in z are in the interior of the unit circle, they will be mapped to the left half s-plane if the approximants $R_{P,P}(s)$, $R_{P,P+1}(s)$ or $R_{P,P+2}(s)$ are used. In other words, it is always possible to find stable transfer functions for the analog filters with a sufficient fitting precision if suitable values for P and Q are chosen.

V. DESIGN EXAMPLES

The prototype digital filter bank is chosen as: $H_0(z) = 1,\ H_1(z) = 1 - z^{-1}$, $F_0(z) = 1 + z^{-1}$, $F_1(z) = -1$. A bandlimited, $|\Omega| \le \pi$, test signal $X(j\Omega) = 1$ is used for the evaluation of signal distortion in the A/D hybrid filter bank. Clearly, we have $H_0(j\Omega) = 1$. The transfer function $H_1(j\Omega)$ has been obtained using the numerical method implemented in the MATLAB instruction INVFREQ(S) [15]. The amplitude and the phase of the output signal for a numerical approximation $L_{2,2}(s)$ are represented in Fig.4. The poles of $H_1(s)$ are at: $s_{1,2} = -2.3885 \pm j2.0579$.

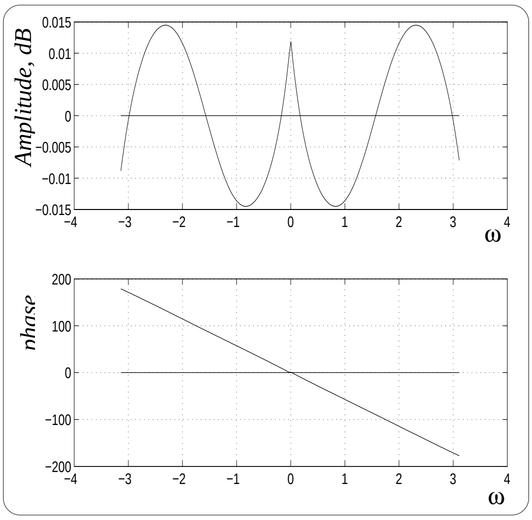

Fig.4. Output amplitude and phase of the two-channel A/D with a numerical approximation $L_{2,2}(s)$.

In the case of D/A system, $F_0(j\Omega)$ has been obtained by the numerical method and $F_1(j\Omega) = -1$. The test signal is $X(e^{j\omega}) = 1$. The amplitude and the phase of the output signal for a numerical approximation $L_{2,2}(s)$ are illustrated in Fig.5. The poles of $F_0(s)$ are the same: $s_{1,2} = -2.3885 \pm j2.0579$.

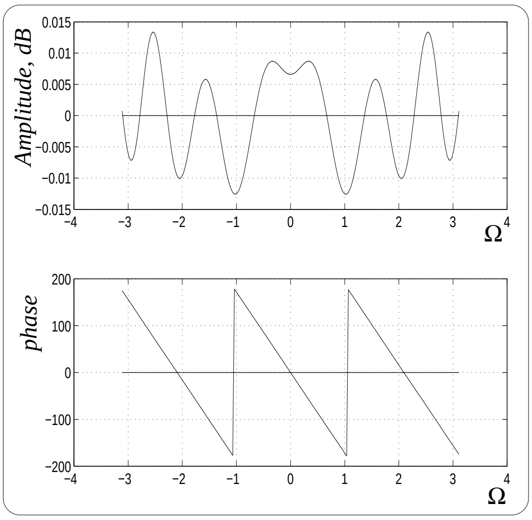

Fig.5. Output amplitude and phase of the two-channel D/A with a numerical approximation $L_{2,2}(s)$.

VI. CONCLUSION

In hybrid filter banks, the perfect reconstruction is achieved only asymptotically. In the proposed design procedure, the analog filters are derived from the digital filters in a prototype digital filter bank. Numerical or analytical methods may be used to obtain the analog filters transfer function. Padé approximation is an analytical technique which was used in this paper to show the existence of stable analog filters. In fact it was shown that if the digital prototype system is designed using FIR or IIR stable filters, stable analog filters can be resulted. However, Padé approximation is not necessarily the best approximation technique in terms of convergence speed. Numerical methods for approximation by complex rational functions can be used to obtain more suitable analog transfer functions. The order of the analog transfer functions depends on both the order of the digital filters and the desired precision of signal reconstruction. The most important application of hybrid filter banks is in A/D and D/A conversion. Three problems must be considered for practical applications of hybrid filter banks: the order of the analog filters, deviation of the analog filters from their nominal characteristics and finally insufficient attenuation of the input signal in the stop-band.

REFERENCES

[1] A. Papoulis, “Generalized sampling expansion”, IEEE Trans. on Circuits and Systems, vol. CAS-24, No. 11, November 1977.

[2] J.L. Brown,” Multi-Channel Sampling of Low-Pass Signals”, IEEE Trans. on Circuits and Systems, vol. CAS-28, NO. 2, February 1981, pp. 101-106.

[3] J.L. Brown,” Multi-Channel Sampling and Reconstruction of Bandlimited Two-Dimensional Signals", IEEE ICASSP, 1988, pp. 1244-1247.

[4] A.R. Figueiras-Vidal, J.B. Marino-Acebal and R.G. Gomez,” On Generalized Sampling Expansions for Deterministic Signals”, IEEE Transactions on Circuits and Systems, vol. CAS-28, NO. 2, February 1981, pp. 153-154.

[5] K.F. Cheung, R.J. Marks, II," Ill-Posed Sampling Theorem", IEEE Trans. on Circtuis and Systems, vol. CAS-32, NO. 5, May 1985, pp.481-484.

[6] J.L. Brown and S.D. Cabrera” On Well-Posedness of the Papoulis Generalized Sampling Expansion”, Trans. on CAS., vol.38, No.5, pp.554-556, May 1991.

[7] J.L. Brown and S.D. Carbrera,”Multi-Channel Signal Reconstruction Using Noisy Samples", IEEE ICASSP, 1990, pp. 1233-1236.

[8] P.P. Vaidyanathan and V. C. Liu,” Classical Sampling Theorems in the Context of Multirate and Polyphase Digital Filter Banks Structures”, IEEE Trans. on Acoustics, Speech and Signal Processing, vol. 36, NO. 9, September 1988, pp. 1480-1495.

[9] J.L. Brown,” Generalized Sampling and the Perfect Reconstruction Problem for Maximally Decimated Filter Banks", IEEE ICASSP, 1989, pp. 1195-1198.

[10] P.P. Vaidyanathan, *Multirate Systems and Filter Banks*, Prentice-Hall, 1993.

[11] G.A. Baker, Jr. and P. Graves-Morris, *Padé Approximants”, PartI and Part II in Encylopedia of Mathematics and its Applications*, volumes 13 and 14, Addison-Wesley Publishing Company, 1981.

[12] E.B. Saff and R.S. Varga,” On the Zeros and Poles of Padé Approximants to e^z“ , Num. Math., 25, 1975, pp.307-322.

[13] R.S. Varga,” On Higher Order Stable Implicit Methods for Solving Parabolic Partial Differential Equations“, J. Math. Phys.,40, 1961, pp.320-231.

[14] B.L. Ehle,” A-Stable Methods and Padé Approximation to the Exponential”, SIAM J. Math. Anal. 4, 1973, pp.671-680

[15] E.C. Levy,” Complex-Curve fitting”, IRE Trans. on Aut. Cont., vol. AC-4, pp.37-44, 1959.