UNDERSAMPLING FOR PARAMETER ESTIMATION WITH APPLICATION TO TIME OF ARRIVAL ESTIMATION

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ABSTRACT

This paper deals with the effect of sampling the continuous observations on parameter estimation errors. In particular, we study the problem of estimating the time of arrival (TOA) of a continuous, deterministic signal in noise. For this problem, the sampling procedure transforms the continuous parameter space into a discrete one, resulting in inherent estimation errors. We introduce a general tool for evaluating the achievable performance for any parameter estimation problem at a given sampling rate. For TOA estimation with a Gaussian-shaped signal, we show that one can undersample with a factor up to 3 times the Nyquist rate with average TOA estimation performance reduction of less than 3dB.

1. INTRODUCTION

Given a continuous process $x(t|\underline{\theta})$, $0 \leq t \leq T$, one is interested in estimating the parameter vector $\underline{\theta}$. In practice, most estimation procedures are implemented by DSP techniques, so the first step is to sample the received data. Indeed, if one samples at the Nyquist rate, it is guaranteed that the achievable estimation performance when using continuous or sampled data is the same. However the Nyquist rate is not necessarily the lowest sampling rate to satisfy this condition. In a given parameter estimation problem, the undersampled data can, at least theoretically, be a sufficient statistic. Moreover, for achieving a given estimation accuracy, one may be able to undersample even more so, resources as system complexity and costs (e.g., memory requirements) can be used more efficiently. In this paper we study the achievable accuracy of a given parameter estimation DSP system as a function of the sampling rate, as well as of the problem limitations (e.g., signal to noise ratio - SNR). We develop a Cramer-Rao type bound on the estimation error. We demonstrate the results by considering the problem of estimating the time of arrival of a deterministic signal in noise. This important problem which is inherent in many practical applications (e.g., [3]) is of particular interest since the sampling introduce quantization error to the estimate.

The paper is organized as follows: In section 2 we introduce the basic idea of how to handle the problem. In section 3 we refer to the specific application of TOA estimation. First we provide an expression for an approximate, Cramer-Rao type bound on the TOA estimation error in a given sampling rate when all the other nuisance parameters are known. Then, we illustrate the results by considering a specific example. We conclude section 3 by discussing the effect of unknown nuisance parameters. The last section is devoted to conclusions and discussion of the results.

2. THE BASIC IDEA

We assume that the continuous process is first filtered by an arbitrary linear filter of a given bandwidth W, and then is sampled at the appropriate Nyquist rate, $f_s = 2W$. We derive the Cramer-Rao lower bound (CRLB) on the estimation error of the parameter vector $\underline{\theta}$ based on the sampled data. This CRLB indicates the achievable estimation performance for $\underline{\theta}$ using the continuous data at the output of the arbitrary filter used, whose impulse response is denoted by $h(t)$. The achievable performance depends on the specific filter used, $h(t)$, as well as on the bandwidth W and on the probability density function (p.d.f) of the received data, $f(x(t)|\underline{\theta})$.

To derive a bound on the achievable performance at a given sampling rate $f_s = 2W$ and on $f(x(t)|\underline{\theta})$ only, we need to eliminate the dependence of the resulting CRLB on $h(t)$. To do so, we minimize the CRLB with respect to $h(t)$ over the space of all linear filters with a given bandwidth W, Ω_W. That is,

$$\begin{aligned} LB(\underline{\theta}|f,W) &= \min_{h(t) \in \Omega_W} CRLB(\underline{\theta}|f,h(t)) \\ &= CRLB(\underline{\theta}|f,h_0(t)) \end{aligned} \quad (1)$$

where we use the notation $LB(\underline{\theta}|f,W)$ for a lower bound on the mean-square error (mse) of any unbiased estimate on $\underline{\theta}$ from the data $x(t|\underline{\theta})$ of p.d.f. $f(x(t)|\underline{\theta})$ using a DSP system with a sampling rate $f_s = 2W$.

Note that while the bound $LB(\underline{\theta}|f,W)$ is independent of the estimation procedure used, it is theoretically achievable (under asymptotic conditions) by the maximum likelihood estimator (MLE) [7] which is applied to the signal at the output of the linear filter $h_0(t) \in \Omega_W$. Applying the MLE to the output of any other linear filter $h(t) \in \Omega_W$ results in mse which is not smaller than $LB(\underline{\theta}|f,W)$. That is, in general:

$$Cov(\hat{\underline{\theta}}|_{h(t)\in\Omega_W}) \geq Cov(\hat{\underline{\theta}}|_{h_0(t)\in\Omega_W}) \geq LB(\underline{\theta}|f,W) \quad (2)$$

The right hand equality holds for the case where the MLE at the output of the filter $h(t) = h_0(t)$ is applied. That is, while developing a lower bound on the achievable estimation procedure in a given sampling rate $f_s = 2W$, we also identify an optimal DSP estimation procedure. This procedure is not unique since the MLE is not the only possible DSP estimation procedure, but it suggests that $h_0(t) \in \Omega_W$ is the optimal pre-filter for any DSP estimation procedure.

Note, however, that (2) holds for a given $W \leq W_0$, where W_0 is the bandwidth of the original continuous signal $x(t)$. The bound on the r.h.s. of (2) can be broken by enlarging W up to W_0, assuming all other statistical factors (e.g., SNR) are kept the same.

Solving the minimization problem in (1) to find $LB(\underline{\theta}|f, W)$ in a given problem is not an easy task. In [1] we have been able to do so for the problem of estimating the arrival time of a step-like signal in white, Gaussian noise and we have presented a close-form simple expression for LB of (1). In this special case, the bandwidth of the continuous data is theoretically infinite, so adding a finite bandwidth constraint is not for reducing complexity, but a practical must. Also, we have found that the optimal pre-filter $h_0(t)$ which is the Canny's filter [2], is very well approximated by the "mother wavelet" used for optimal sampling in the discrete wavelet transform (e.g., [5]).

3. APPLICATION: ESTIMATING THE TIME OF ARRIVAL OF A DETERMINISTIC SIGNAL

As in the classical modeling of the TOA estimation problem [3,4,6], we assume a signal $s(t|\psi)$ which is known up to a set of p parameters (e.g., its amplitude), represented by ψ. The problem is to estimate its time of arrival (TOA), t_0, from:

$$x(t) = s(t - t_0|\psi) + n(t) \quad , \quad 0 \leq t \leq T \tag{3}$$

where $n(t)$ is a zero mean, stationary Gaussian noise and $s(t)$ is a deterministic signal of bandwidth W_0. The parameter vector to be estimated is $\underline{\theta} = [t_0, \psi]^T$. We first consider the case where the nuisance parameters vector ψ is known.

The CRLB for the estimation error of $\hat{t_0}$ from $x(t)$ of (3) satisfies [4]:

$$Var\{\hat{t_0}\} \geq CRLB(\hat{t_0}|s, \psi, f) = \frac{\frac{N_0}{2}}{\int_0^T (\frac{ds(t|\psi)}{dt})^2 dt} \tag{4}$$

where $N_0 = \frac{\sigma_n^2}{W_0}$ is the spectral density of the assumed white noise process.

We proceed by assuming that the received signal $x(t)$ of (3) is pre-filtered by a linear filter of impulse response $h_s(t) = \frac{1}{s}h(\frac{t}{s})$. The bandwidth of $h(t)$ is normalized to unity so the scalar $2/s$ is the bandwidth of the filter $h_s(t)$, previously denoted by W. The output of the filter is then sampled at the Nyquist sampling rate $\frac{1}{s}$. The $N = \frac{T}{s}$ samples are put in an N-dimensional vector $\underline{y}$ which is a Gaussian vector of mean $\underline{\mu}$ and covariance $N_0 C$, where:

$$\underline{\mu} = x(t - t_0) * h_s(t)|_{t=ns+\tau}$$
$$and$$
$$C_{ij} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} r_n(u - v)h_s(u + is)h_s(v + js)dudv$$
$$i, j = 1, ..., N$$

where * stands for the convolution operation, $r_n(\cdot) = R_n(\cdot)/\sigma_n^2$ is the normalized noise correlation function and $s \geq \tau = \lfloor t_0/s \rfloor \geq 0$ presents the synchronization factor in the sampling operation. For white noise with $r_n(u) = \delta(u)$, $C_{ij} = \int_{-\infty}^{\infty} h_s(u + is)h_s(u + js)du \quad i, j = 1, ..., N$. Direct derivation of $I = -E\{\frac{\partial^2 lnf(\underline{y}|\theta)}{\partial\theta^2}\}$ yields:

$$I(t_0, \tau, s, h) = \frac{1}{N_0}\underline{w}^T C^{-1}\underline{w} \tag{5}$$

where $\underline{w}$ is an N-dimensional vector whose n-th entry is:

$$\begin{aligned} w_n &= \frac{\partial[s(t - t_0) * h_s(t)]}{\partial t_0}|_{t=ns+\tau} \\ &= s(t) * \frac{\partial h_s(t - t_0)}{\partial t_0}|_{t=ns+\tau} = -\frac{1}{s}W_{sf}(t - t_0)|_{t=ns+\tau} \end{aligned}$$

Note that $W_{sf}(t) = s * [sh'(\frac{t}{s})]$ is the wavelet expansion of $s(t)$ with the analyzing function $h'(t) = \frac{dh(t)}{dt}$.

The matrix C is a symmetric, NxN Toeplitz matrix. Denote by $\gamma_i, \quad i = 0, .., N-1$, the element of the i-th diagonal such that $C_{mn} = \gamma_i \quad for \quad |m - n| = i$. We assume that the function $h(t)$ has a finite support so $\gamma_i = 0$ for $i > M$, where, for observation time (T) sufficiently large compared to the noise correlation time, $M << N$. Under this assumption the end effects can be ignored so that for $N \to \infty$ C^{-1} is asymptotically a symmetric, Toeplitz matrix. The inverse matrix elements are denoted by $C_{mn}^{-1} = c_i^{-1}$ for $|m - n| = i$. Equation (5) can then be written as:

$$\begin{aligned} I(t_0, \tau, s, h) \simeq \frac{1}{N_0}\{c_0^{-1}\sum_{n=0}^{N-1}\omega_n^2 + 2c_1^{-1}\sum_{n=0}^{N-2}\omega_n\omega_{n+1} + ... \\ + 2c_{N-1}^{-1}\omega_0\omega_{N-1}\} \end{aligned} \tag{6}$$

The resulting CRLB (the inverse of I) depends on the filter $h(t)$, on τ and on $s = \frac{1}{2W}$. Before maximizing (6) with respect to $h(t)$, we need to eliminate τ, which can be regarded as a synchronization factor which does not appear in asynchronous estimation procedures (such as the matched filter). From a more theoretical point of view, it can regarded as a mismatch between the parameter space and the estimates: it is reasonable to assume that $\underline{\theta} \in R^{P+1}$, where $\underline{\theta} = [t_0, \psi]^T$. Usually, also $\hat{\psi} \in R^P$. However, the sampling operation forces $\hat{t_0}$ to be at a subspace of R. In fact, if t_s is the sampling time then $\hat{t_0}/t_s \in Z$ while $t_0 \in R$. $\tau = (t_0 - \hat{t_0})/t_s \in [0, s]$ represents this difference. It is reasonable to assume that τ is a random variable, uniformly distributed over $[0, s]$. Since the Cramer-Rao inequality is satisfied for any τ, it is also satisfied under averaging over τ, so: $Var\{\hat{t_0}|s\} \cdot I(t_0, \tau, s, h) \geq 1$ yields $Var\{\hat{t_0}|s\} \cdot E_\tau\{I(t_0, \tau, s, h)\} \geq 1$. Therefore (6) yields:

$$Var\{\hat{t_0}|s\} \geq \frac{1}{\frac{1}{s}\int_0^s I(t_0, \tau, s, h)d\tau} = \bar{I}^{-1}(t_0, s, h) \tag{7}$$

where $\bar{I}(t_0, s, h) = \frac{1}{s}\int_0^s I(t_0, \tau, s, h)d\tau$.

Applying (7) on (6) yields:

$$\bar{I}(t_0,\tau,s,h) \simeq \frac{1}{N_0}\{c_0^{-1}\int_0^T \omega_0^2 d\tau + 2c_1^{-1}\int_0^T \omega_0\omega_1 d\tau + ... + 2c_{N-1}^{-1}\int_0^T \omega_0\omega_{N-1}d\tau\} \qquad (8)$$

Note that both the coefficients c_n^{-1} which are related to the covariance of the sampled noise and the signal wavelet coefficients $w_n = -\frac{1}{s}W_{sf}(t-t_0)|_{t=ns+\tau}$ depend on $h(t)$. The optimal choice of $h(t)$ which maximizes $\bar{I}$ of (8) therefore depends on both $s(t)$ and on the noise covariance function. However, from our experience with different waveshapes, $h(t)$ which makes the matrix C close to a diagonal matrix is a good choice, independent of the signal $s(t)$. With a white, Gaussian noise such h(t) is the cubic spline, given by:

$$h(t) = \{ \begin{array}{cl} \frac{4}{3}+8|t|^3-8t^2 & ;\ 0 \le |t| \le 0.5 \\ \frac{8}{3}(1-|t|)^3 & ;\ 0.5 < |t| \le 1 \\ 0 & ;\ elsewhere \end{array} . \qquad (9)$$

That is, the mother wavelet used for calculating $\{\omega_n\}$ is the derivative of the cubic spline - the quadratic spline [5].

Under this choice, $c_i^{-1} \approx 0$ for $i \neq 0$. Therefore, the bound of (2) for the estimation error of TOA of a known signal in zero mean, white Gaussian noise at a given sampling rate $f_s = 1/s$ can be approximated by:

$$ALB(t_0|N_0,s) \approx \left[\frac{1}{s^2 N_0 \gamma}\int_{-T}^{T}[W_s(u-t_0)]^2 du\right]^{-1} \qquad (10)$$

where $W_s(t) = s(t) * [\frac{1}{s}h'(\frac{t}{s})]$ and $\gamma = \int_{-1}^{1} h^2(t)dt$, for $h(t)$ given by (9).

3.1. Example

Assume a signal $s(t|\psi)$ which has a modulated Gaussian shape:

$$s(t|\psi) = Ae^{-(2\pi f_1 t)^2}sin(2\pi f_2 t + \phi) \qquad (11)$$

Such a signal characterizes ultrasonic measurements. For example, in the data we were given, typical parameters are: $A = 100\mu V; f_1 = 5MHz; f_2 = 10MHz$. The delay t_0 is typically not greater than $200\mu Sec$. It can be shown that the CRLB for estimating the TOA of $s(t)$ from the continuous data $x(t)$ of (3), where all the nuisance parameters $\psi = (A, f_1, f_2, \phi)$ are known, can be approximated by:

$$CRLB(t_0|\psi) \approx \frac{3.77}{\sqrt{\pi}}\frac{N_0}{f_1 A^2}. \qquad (12)$$

We compare this bound with the achievable estimation performance under undersampling conditions, as described by the ALB of (10). In Fig. 1 we depict the CRLB of (12) and the approximated LB of (10) (ALB) which has been derived numerically for $s(t)$ of (11) and $h(t)$ of (9). Actually, the wavelet expansion coefficients W_s have been evaluated using Matlab routines for the fast wavelet transform. The two bounds are presented as a function of the sampling rate, normalized by the Nyquist rate, $f_s = 2W_0$, for $N_0 = 1$. For the Gaussian waveshape we assumed that the bandwidth of

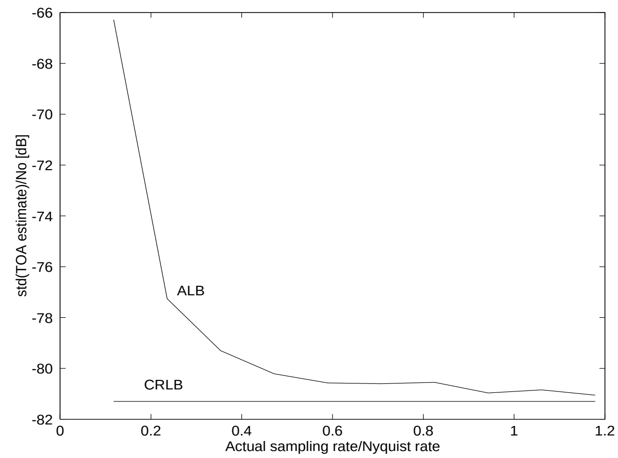

Figure 1: The estimation error of t_0 as a function of the sampling rate and the corresponding CRLB

the signal is the quantity corresponding to three standard deviations, i.e., $W_0 = 3\sqrt{2}f_1 Hz$.

This example demonstrates the general characteristics of the achievable TOA estimation performance. Above the Nyquist rate, increasing the sampling rate (oversampling) does not improve performance. Below this point, decreasing the sampling rate (undersampling) decreases the performance. However, in this example one can undersample by a factor of 3 while losing only about 3dB in the estimation performance. Undersampling by a larger factor causes a significant degradation in the achievable estimation performance.

3.2. Unknown nuisance parameters

If the nuisance parameters ψ are unknown, the entire vector $\underline{\theta} = (t_0, \psi)$ should be estimated. To carry out the approach of section 2, we now need to do the following:

i) Construct the Fisher information matrix (FIM) J whose (i,j) entry is given by: $J_{ij} = -E\{\frac{\partial^2 ln f(\underline{y}|\underline{\theta})}{\partial\theta_i\partial\theta_j}\}$ for i,j=1,...p+1. $\underline{y}$ is the vector of the samples at the output of the filter h, as described above equation (5). The scalar I of (5) is the (1,1) entry of the resulting FIM.

ii) Average the resulting (p+1)x(p+1) FIM with respect to τ. The (1,1) entry of the resulting matrix, $\bar{J} = E_\tau\{J\}$, is $\bar{I}$ of (7).

iii) Maximize the resulting matrix, $\bar{J}(f,s,\underline{\theta},h)$, with respect to h. The inverse of the matrix $\bar{J}$ at $h = h_0$ is the Cramer-Rao matrix whose diagonal entries bound the estimation errors of $\underline{\theta}$.

The last step is not well defined. Different approaches can be taken with respect to the required maximization. Two are listed below:

- Find h_o such that $\bar{J}(f,s,\underline{\theta},h) - \bar{J}(f,s,\underline{\theta},h_0)$ is a non negative definite matrix for any s and $\underline{\theta}$.

- Find h_0 which minimizes the (1,1) entry of the inverse of $\bar{J}$ which is the bound on t_0. That is, maximizes the scalar $\bar{J}_{11} - \bar{J}_{t_0\psi}\bar{J}_{\psi\psi}^{-1}\bar{J}_{\psi t_0}$ with respect to h(t).

The first option means to optimize the FIM with respect to all unknown parameters. The second looks for the optimal choice for t_0, which may not be optimal for the nuisance parameters ψ. Note that the first condition is sufficient but not necessary for the second condition to hold.

The optimal way for performing the maximization is still under study. However, from a practical point of view, for TOA estimation choosing h of (9) is satisfactory even for the case of unknown nuisance parameters.

4. CONCLUSIONS

Is the Nyquist sampling rate necessary for optimal parameter estimation from continuous data? Obviously, it is sufficient. However, this paper suggests that one can undersample without paying in estimation performance.

We have presented a general tool to evaluate the achievable parameter estimation performance using undersampled data. Using this tool one can evaluate the amount of parameter estimation performance degradation corresponding with a certain undersampling factor. We have applied this tool to the classic problem of time of arrival estimation and, in a certain example we have shown that the performance loss due to undersampling by a factor of 3 or less is bounded by 3dB. This example show that one may consider undersampling as a way to reduce complexity of parameter estimation DSP systems.

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