A BLOCK-ITERATIVE QUADRATIC SIGNAL RECOVERY ALGORITHM

Patrick L. Combettes

Department of Electrical Engineering
City College and Graduate School
City University of New York
New York, NY 10031, USA

ABSTRACT

We propose a block-iterative parallel decomposition method to solve quadratic signal recovery problems under convex constraints. The idea of the method is to disintegrate the original multi-constraint problem into a sequence of simple quadratic minimizations over the intersection of two half-spaces constructed by linearizing blocks of constraints. The implementation of the algorithm is quite flexible thanks to its block-parallel structure. In addition a wide range of complex constraints can be incorporated since the method does not require exact constraint enforcement at each step but merely approximate enforcement via linearization. An application to deconvolution is demonstrated.

1. INTRODUCTION

The convex set theoretic signal recovery problem is to produce an estimate of an original signal in the intersection of a family $(S_i)_{1\leq i\leq m}$ of closed and convex sets in a real Hilbert space Ξ [4], [12], [13]. The recovery problem therefore reads

$$\text{Find } a^\star \in S = \bigcap_{i=1}^{m} S_i. \tag{1}$$

This convex feasibility framework has been applied to numerous signal recovery problems (see, e.g., [4] and the references therein). Each set S_i represents the class of signals that satisfy a certain constraint and is typically given as

$$S_i = \{a \in \Xi \mid g_i(a) \leq 0\}, \tag{2}$$

where $g_i : \Xi \to \mathbb{R}$ is a continuous convex function [5].

In some instances, it is more appropriate to select the feasible signal that is closest to a reference signal r as opposed to *any* signal in S, e.g., [2], [3], [9]. The recovery problem is then to find the signal $a^* \in S$ that least deviates from r relative to the underlying norm, i.e., to solve the quadratic program

$$\begin{cases} \min\ \frac{1}{2}\|a-r\|^2 \\ \text{subject to} \\ a \in S = \bigcap_{i=1}^{m} S_i. \end{cases} \tag{3}$$

In other words, one seeks the projection a^* of r onto the feasibility set S (the minimum norm solution for $r = 0$).

In the signal recovery and mathematical programming literatures, several methods can be found that solve (3) under certain conditions. From a numerical point of view, however, these methods suffer from several limitations and they can seldom be implemented efficiently. In this paper we propose a new constrained quadratic minimization method that alleviates these limitations and is particularly well suited for large-scale signal recovery applications in the presence of complex convex constraints. The method is block-parallel and it can therefore fully take advantage of parallel processing architectures. Its computational efficiency is further enhanced by the fact that it does not require exact enforcement of the constraints but merely approximate enforcement by means of linearizations.

The remainder of the paper is divided in three sections. In Section 2, we review existing quadratic minimization methods and discuss their limitations in the context of signal recovery applications. The new method is presented in Section 3. Finally, a numerical application to signal deconvolution in spectroscopy is demonstrated in Section 4.

2. REVIEW OF EXISTING METHODS

Several algorithms have been proposed in the signal/image recovery literature, that can solve (3) in specific cases. Thus, the algorithm of [10] is restricted to a nonnegativity constraint while the approach of [7] provides a finite parametrization for linear data models

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that it is limited in practice to simple constraints, as projections onto the feasibility set are required. The method of [9] is restricted to $m = 2$ sets and is relatively involved. In the quadratic programming literature, algorithms have been proposed for certain types of constraint sets such as half-spaces, cones, affine subspaces, hyperslabs (see [3] and references therein).

A popular algorithm for solving (3) under our general assumptions is that given in [1]. This so-called Dykstra-projection method proceeds via the cyclic iteration scheme

$$\begin{cases} a_0 = r \\ \text{and} \\ (\forall n \in \mathbb{N}) \quad a_{n+1} = \overset{\vee}{P}_{n\,(\text{modulo } m)+1}(a_n). \end{cases} \tag{4}$$

Here $\overset{\vee}{P}_i$ is a "rectified projection" onto S_i, i.e., $\overset{\vee}{P}_i(a_n) = P_i(a_n + b_{i,n})$, where $b_{i,n}$ is the outward normal of S_i at the previous projection onto S_i, obtained m iterations earlier, and P_i the projection operator onto S_i.[1] The parallel version

$$\begin{cases} a_0 = r \\ \text{and} \\ (\forall n \in \mathbb{N}) \quad a_{n+1} = \dfrac{1}{m}\displaystyle\sum_{i=1}^{m} \overset{\vee}{P}_i(a_n) \end{cases} \tag{5}$$

was introduced in [8]. An alternative parallel method, proposed in [4], is described by the recursion

$$\begin{cases} a_0 = r \\ \text{and} \\ (\forall n \in \mathbb{N}) \quad a_{n+1} = \dfrac{1}{n+1} a_0 + \dfrac{n}{(n+1)m}\displaystyle\sum_{i=1}^{m} P_i(a_n) \end{cases} \tag{6}$$

(a serial version is also available [4]). Despite their conceptual generality, these methods face several obstacles in actual signal recovery applications.

1. Signal recovery problems are often large-scale problems (e.g., image or three-dimensional applications). It is therefore of utmost importance that the algorithms be implementable in a flexible manner on a parallel architecture. Algorithm (4) is serial and therefore ill-suited for parallel processing. Algorithms (5) and (6) are fully parallel as they require that *all* the sets be activated at each iteration. As a result, if the number of sets exceeds the number of concurrent processors available, the implementation will not be optimal.

2. Algorithms (4) and (5) require the storage of auxiliary vectors as one outer normal per set must be carried from one iteration to the next. This complicates the implementation of these algorithms in terms of memory allocation and management.

3. Algorithms (4), (5), and (6) all require that the projection operators $(P_i)_{1 \le i \le m}$ onto the sets be known. As discussed in [4] and [5], there are many useful constraints for which projections are not available in closed-form and must be computed iteratively as costly subproblems.

4. In numerical tests, algorithms (4), (5), and (6) have been observed to converge slowly.

3. THE PROPOSED ALGORITHM

Principle. Starting with $a_0 = r$, the proposed method proceeds as follows. At iteration n, one selects a block of indices $I_n \subset \{1, \ldots, m\}$ and computes *approximate* projections $(p_{i,n})_{i \in I_n}$ of the current iterate a_n onto the sets $(S_i)_{i \in I_n}$. Given (2), these approximate projections are efficiently computed as subgradient projections, i.e., [5]

$$p_{i,n} = \begin{cases} a_n - \dfrac{g_i(a_n)}{\|t_{i,n}\|^2}\, t_{i,n} & \text{if } a_n \notin S_i \\ a_n & \text{otherwise,} \end{cases} \tag{7}$$

where $t_{i,n}$ is a subgradient (the gradient if g_i is differentiable) of g_i at a_n.[2] One then forms a relaxed convex combination $a_{n+\frac{1}{2}}$ of these subgradient projections and constructs the two half-spaces

$$\begin{cases} H_n^1 = \{a \in \Xi \mid \langle a - a_{n+\frac{1}{2}} \mid a_n - a_{n+\frac{1}{2}} \rangle \le 0\} \\ H_n^2 = \{a \in \Xi \mid \langle a - a_n \mid a_0 - a_n \rangle \le 0\}. \end{cases} \tag{8}$$

The set $H_n^1 \cap H_n^2$ serves as an outer approximation to the feasibility set S and the new iterate a_{n+1} is obtained as the projection of a_0 onto $H_n^1 \cap H_n^2$. We now present the detailed algorithm (the relaxed average of subgradient projections $a_{n+\frac{1}{2}}$ is computed at step S2 and the projection of a_0 onto $H_n^1 \cap H_n^2$ at step S3.)

Algorithm.[3] A sequence $(a_n)_{n \ge 0}$ is constructed as follows.

S0. Fix $(\delta, \varepsilon) \in]0, 1[^2$ and set $a_0 = r$ and $n = 0$.

S1. Choose a block of indices $I_n \subset \{1, \ldots, m\}$.

[1] The projection of $a \in \Xi$ onto S_i is the unique point $P_i(a) \in S_i$ such that $\|P_i(a) - a\| = \inf_{b \in S_i} \|b - a\|$ [4].

[2] The reader is referred to [5] for a tutorial overview of subgradients and signal recovery examples. In that paper, subgradient projections were used to solve (1).

[3] One recovers a method proposed in [11] as a special case by taking exact projections as opposed to approximate ones, letting $I_n = \{1, \ldots, m\}$, and $w_{i,n} = 1/m$.

S2. Set

$$a_{n+\frac{1}{2}} = a_n + \lambda_n \left(\sum_{i \in I_n} w_{i,n} p_{i,n} - a_n \right), \qquad (9)$$

where

(A) For every $i \in I_n$, $p_{i,n}$ is as in (7).

(B) $(w_{i,n})_{i \in I_n} \subset]\delta, 1]$ and $\sum_{i \in I_n} w_{i,n} = 1$.

(C) $\varepsilon L_n \leq \lambda_n \leq L_n$ where

$$L_n = \begin{cases} \dfrac{\sum_{i \in I_n} w_{i,n} \|p_{i,n} - a_n\|^2}{\left\| \sum_{i \in I_n} w_{i,n} p_{i,n} - a_n \right\|^2} & \text{if } a_n \notin \bigcap_{i \in I_n} S_{i,n} \\ 1 & \text{else.} \end{cases}$$

S3. Set $\pi_n = \langle a_0 - a_n \mid a_n - a_{n+\frac{1}{2}} \rangle$, $\mu_n = \|a_0 - a_n\|^2$, $\nu_n = \|a_n - a_{n+\frac{1}{2}}\|^2$, $\rho_n = \mu_n \nu_n - \pi_n^2$, and

$$a_{n+1} = \begin{cases} a_{n+\frac{1}{2}} \text{ if } \rho_n = 0 \text{ and } \pi_n \geq 0, \\ a_0 + (1 + \pi_n / \nu_n)(a_{n+\frac{1}{2}} - a_n) \\ \qquad \text{if } \rho_n > 0 \text{ and } \pi_n \nu_n \geq \rho_n, \\ a_n + \dfrac{\nu_n}{\rho_n} (\pi_n (a_0 - a_n) - \mu_n (a_n - a_{n+\frac{1}{2}})) \\ \qquad \text{if } \rho_n > 0 \text{ and } \pi_n \nu_n < \rho_n. \end{cases}$$

S4. Set $n = n + 1$ and go to S1.

Convergence. [6] Suppose that the two conditions below are fulfilled:

(i) The control sequence $(I_n)_{n \geq 0}$ is such that every set is activated at least once within any M consecutive iterations: there exists a positive integer M such that

$$(\forall n \in \mathbb{N}) \ \ \{1, \ldots, m\} = \bigcup_{k=0}^{M-1} I_{n+k}. \qquad (10)$$

(ii) The subgradients of the functions $(g_i)_{1 \leq i \leq m}$ are uniformly bounded on bounded sets: for every $\gamma \in]0, +\infty[$ there exists $\zeta \in]0, +\infty[$ such that the condition $\|a\| \leq \gamma$ implies that for every $i \in \{1, \ldots, m\}$ and every subgradient t of g_i at a we have $\|t\| \leq \zeta$.[4]

Then every sequence $(a_n)_{n \geq 0}$ generated by the algorithm converges strongly to the solution $a^\star$ of (3).

Comments. The proposed algorithm displays three salient features that make it very attractive in comparison with existing schemes.

1. It is block-iterative and parallel: only the constraints with indices in I_n (not *all*) are activated simultaneously at iteration n. The control rule (10) offers in this respect great flexibility in the selection of the constraints.
2. It requires only subgradient projections as opposed to exact ones. Analytically complex constraints can therefore be incorporated in the recovery algorithm and processed at low cost.
3. Initial numerical experiments indicate that it displays a fast speed of convergence.

4. NUMERICAL EXAMPLE

We consider a spectroscopy problem described in [12], where it was solved as a feasibility problem, i.e., (1).

The 64-point original emission spectrum h shown in Fig. 1(A) has been degraded by the limited resolution of the spectrometer and recording noise u. The recorded signal $x = Lh + u$ (the matrix L models the blurring induced by the spectrometer) is shown in Fig. 1(B). The feasibility set S is defined by the following constraints (see [12] for details): nonnegativity of h, a residual/noise variance matching constraint, and residual/noise amplitude matching constraints.

We seek the feasible signal a^* that least deviates from the recorded signal, i.e., $r = x$ in (3). The solution a^* to this problem is shown in Fig. 1(C). Next, we show the restorations obtained after 30 iterations of three parallel algorithms: algorithm (5) in Fig. 1(D); algorithm (6) in Fig. 1(E); the proposed algorithm in Fig. 1(F). This experiment shows that proposed algorithm converges much faster than (5) and (6) in terms of iteration counts. Furthermore, (5) and (6) must use exact projections. In the case of the variance matching constraint set, which takes the form

$$S_2 = \{a \in \mathbb{R}^{64} \mid \|x - La\|^2 \leq \zeta\}, \qquad (11)$$

it was shown in [12] that the projection $P_2(a_n)$ is not known in closed form and must be calculated via a costly multi-step procedure. By contrast, with the proposed algorithm, it can be approximated by the subgradient projection given by (7) for $a_n \notin S_2$ as

$$p_{2,n} = a_n + \frac{\|r_n\|^2 - \zeta}{2 \, \| {}^tLr_n\|^2} \, {}^tLr_n \qquad (12)$$

where ${}^tL$ is the transpose of L and $r_n = x - La_n$ [5]. Let us stress that evaluating $p_{2,n}$ is a one-step procedure which is about 15 times cheaper than the multi-step procedure proposed in [12] to evaluate $P_2(a_n)$. This further reduces the computational load of the proposed algorithm vis-à-vis (5) and (6).

[4] This standard condition is automatically satisfied in finite dimensional spaces [5].

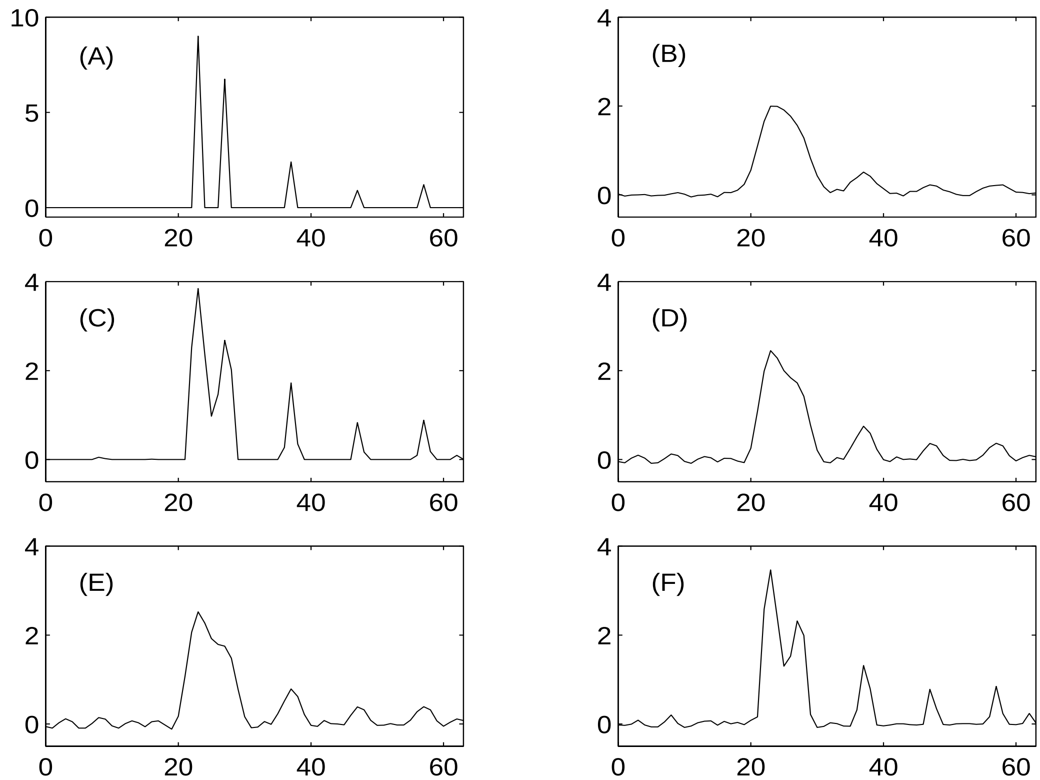

Figure 1: Constrained quadratic recovery of a spectrogram.

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