# THE EFFECT OF SAMPLING AND QUANTIZATION ON FREQUENCY ESTIMATION


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**ABSTRACT**

The effect of sampling and quantization on frequency estimation for a single sinusoid is investigated. Cramér-Rao bound for 1 bit quantization is derived, and compared with the limit of infinite quantization. It is found that 1 bit quantization gives a slightly worse performance, however, with a dramatic increase of variance at certain frequencies. This can be avoided by using 4 times oversampling. The effect of sampling when using non-ideal anti-aliasing lowpass filters is therefore investigated. Cramér-Rao lower bounds are derived, and the optimal filters and sampling frequencies are found. Finally, fast estimators for 1 bit sampling, in particular correlation based estimators, are derived. The paper is concluded with simulation results for 4 times oversampled 1 bit quantization.


## 1. INTRODUCTION

We consider the classical problem of estimating the frequency, phase and amplitude of a *single* complex sinusoid in additive, white Gaussian noise. Thus, the continuous time observed signal is

$$\begin{aligned} x(t) &= s(t;\theta) + v(t), \qquad t \in (-\infty, \infty) \\ s(t;\theta) &= Ae^{i(\omega t+\phi)} \end{aligned} \tag{1}$$

where $v(t)$ is continuous time white Gaussian noise (WGN) with power $\sigma^2$, $A$, $\theta = [\omega\ \phi]^T$ are the unknown parameters, and $\omega \in (-\pi, \pi]$.

Usually, the signal is processed digitally. In order to do the digital processing, the signal must be sampled and quantized. In most cases, analyses of accuracy do not consider this process, although it can considerably influence the accuracy. In this paper we will consider the influence of sampling and quantization, and in particular optimization of sampling and quantization with respect to accuracy and in a trade off with complexity.

Prior to sampling, the signal is transmitted through an analog anti-aliasing filter. We here assume that the signal has been stationary for so long time prior to the start of the sampling process (or that it has a smooth envelope), that we can disregard the transient response. Thus, if the antialising filter has frequency response $H(\omega)$ and the sampling time is $T_s$ the sampled signal is[1]

$$\begin{aligned} x[k] &= s[k;\theta] + v[k], \qquad k = 0, \dots, N-1 \\ s[k;\theta] &= AH(\omega)e^{i(\omega T_s k+\phi)}. \end{aligned} \tag{2}$$


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[1] We denote all quantities associated with the continuous time signal by $(\cdot)$ and all quantities associated with the discrete time signal by $[\cdot]$.

Here $v[k]$ is additive Gaussian noise, which is not necessarily white. We will return to the specific characteristics of the discrete time noise below.

After sampling, the signal is quantized, i.e., rounded to one of a finite number of levels. If the quantization is very fine, e.g., 12 bits precision, the quantization can be disregarded or treated as another source of additive noise. However, some applications deal with very high frequency signals (Giga Hertz range) and fine quantization is impossibly or economically infeasible. We therefore consider coarse quantization, in particular single-bit quantization, with a signal given by,

$$x_+[k] = \mathrm{sign}(\Re(x[k])) + i\,\mathrm{sign}(\Im(x[k])), k = 0, \dots, N-1. \tag{3}$$

Apart from being simple to implement, 1 bit quantization also has the advantage that no gain control is needed and, as we will see below, that very efficient algorithms for processing of one bit samples can be made.

The classical case is the case with infinitely fine quantization, ideal low-pass antialising filters, and $T_s = 1$. In this case, the Fisher information matrix for the unknown parameters $\theta = [\omega\ \phi]^T$ is given by [4]

$$\mathbf{I}(\theta) = 2\left(\frac{A}{\sigma}\right)^2 \sum_{k=0}^{N-1} \begin{bmatrix} k^2 & k \\ k & 1 \end{bmatrix}, \tag{4}$$

which can be explicitly summed and inverted to give the Cramér-Rao bound (CRB)

$$\mathrm{CRB}(\hat{\omega}) = \frac{6\sigma^2}{A^2 N(N^2-1)}. \tag{5}$$

Furthermore, the maximum likelihood estimator (MLE) can be implemented by finding the maximum peak of the DFT of $x[k]$ [4].

## 2. CRAMÉR-RAO BOUND FOR ONE-BIT QUANTIZATION

At first we will consider the case of ideal low-pass anti-aliasing filters and one bit quantization. Let $r[k] = \Re(x_+[k])$ and $s[k] = \Im(x_+[k])$. The pdf of $r[k]$ is then given by

$$f_{r[k]}(q;\theta) = \frac{1}{\sqrt{\pi}\sigma}\int_0^\infty \exp\left(-\frac{1}{\sigma^2}(x - qA\cos(\omega n+\phi))^2\right)dx \tag{6}$$

where $q = \pm 1$. We get a similar expression for the pdf of $s[k]$, just with sin instead of cos. Notice that this pdf is (continuously)

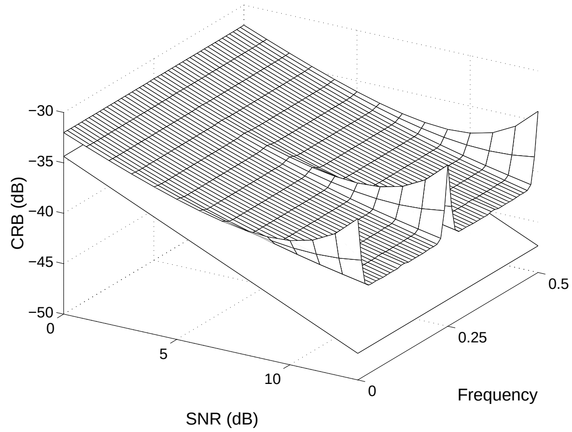


Figure 1: CRB as a function of frequency ($\frac{\omega}{2\pi}$) and SNR ($10\log(\frac{2A^2}{\sigma^2})$) for $N = 32$ samples. The lower surface is for no quantization, while the upper surface is for 1 bit quantization.

differentiable with respect to $\theta$, and the CRB does therefore indeed exist. Since the variables $r[k]$ are i.i.d. the Fisher matrix due to $r[k]$ can be found as

$$\mathbf{I}^r(\theta) = \sum_{k=1}^{N} \sum_{q=\pm 1} \frac{1}{f_{r[k]}(q;\theta)} \frac{\partial f_{r[k]}}{\partial \theta}(q;\theta) \frac{\partial f_{r[k]}}{\partial \theta}(q;\theta)^T, \quad (7)$$

with a similar expression for $\mathbf{I}^s(\theta)$ due to $s[k]$. Since $r[k]$ and $s[k]$ are uncorrelated, the total Fisher matrix is

$$\mathbf{I}(\theta) = \mathbf{I}^r(\theta) + \mathbf{I}^s(\theta). \quad (8)$$

Calculating this with the help of Maple, we get the following expression when the unknown parameters are $\theta = [\omega\ \phi]^T$

$$\mathbf{I}(\theta) = 2\frac{2}{\pi}\left(\frac{A}{\sigma}\right)^2 \sum_{k=0}^{N-1} \begin{bmatrix} k^2 & k \\ k & 1 \end{bmatrix} \kappa(\omega k + \phi; A, \sigma) \quad (9)$$

where

$$\begin{aligned} \kappa(\varphi; A, \sigma) \quad = \quad & \frac{\exp\left(-2\left(\frac{A}{\sigma}\right)^2 \cos^2\varphi\right)}{1 - \operatorname{erf}^2\left(\frac{A}{\sigma}\cos\varphi\right)} \sin^2\varphi \\ & + \frac{\exp\left(-2\left(\frac{A}{\sigma}\right)^2 \sin^2\varphi\right)}{1 - \operatorname{erf}^2\left(\frac{A}{\sigma}\sin\varphi\right)} \cos^2\varphi. \quad (10) \end{aligned}$$

In contrast to the ideal case, the Fisher matrix and CRB cannot be summed explicitly to give a closed form formula. However, when comparing with (4) we can still reach some conclusions. The formulas differ by the factor $\frac{2}{\pi}$ and the function $\kappa(\varphi; A, \sigma)$. It can be proven that $\kappa(\varphi; A, \sigma) < 1$ so that we can conclude that the CRB for 1 bit quantization is at least $\frac{\pi}{2} \approx 1.6$ larger than for no quantization. Furthermore $\kappa(\varphi; A, \sigma)$ is strongly varying with $\varphi$ with a periodicity of $\frac{\pi}{4}$, so that we can also expect the CRB to be strongly dependent on frequency and phase with a periodicity of $\frac{\pi}{4}$.

Figure 1 shows a plot of the CRB for $N = 32$ samples as a function of frequency and SNR. The CRB is also dependent on the phase, but as it is a reasonable assumption that the phase is a uniform random variable, the CRB has been averaged over the phase.

As can be seen the CRB has a catastrophic increase around multiples of $\frac{\pi}{4}$ of $\omega$, where in fact the variance *increases* with increasing SNR. The only way to avoid these critical frequencies is to make sure that the frequencies of interest are between these critical frequencies, i.e., by oversampling the continuous time signal by at least a factor of 4.

## 3. SAMPLING OF CONTINUOUS TIME POLE-ZERO FILTERS

As was shown in the previous section, when using coarse quantization it is necessary to oversample the signal in order to avoid the variance increase at certain frequencies. It is therefore also necessary to consider the actual anti-aliasing filters used prior to sampling. Here we will only consider Butterworth filters [2] of order $n$ with

$$S(s) = H(s)H(-s^*)^* = \frac{1}{1 + \left(\frac{s}{2\pi f_c}\right)^{2n}}. \quad (11)$$

The continuous time WGN process $v(t)$ is transmitted through the filter $H(s)$, with output $v'(t)$. The process $v'(t)$ then is a Gaussian random process with spectrum $S(s)$ and correlation function $R(t)$. Denote the poles of $S(s)$ by $p_1, \ldots, p_{2n}$. The process $v'(t)$ is sampled equidistantly with sampling interval $T_s = \frac{2\pi}{\omega_s}$, giving the discrete time noise $v[k]$. It is clear that the correlation function of $v[k]$ is the sampled correlation function of $v'(t)$, $R[k] = R(kT_s)$. We can therefore find the spectrum of $v[k]$ by residue calculus (or partial fraction expansion),

$$S[z] = \sum_{i=1}^{2n} \operatorname{Res}\left(S(p_i)\frac{1}{1 - \exp(p_i T_s)z^{-1}}\right). \quad (12)$$

We can also model $v[k]$ as a discrete time regular process, by finding the minimum phase factor $H[z]$ of $S[z] = H[z]H[1/z^*]^*$, i.e., isolating the zeros inside the unit-circle. We hereby find that $v[n]$ is an ARMA$(n, n-1)$ process. Except for order 1, it is impossible to find closed form formulas, but the ARMA coefficients can be found numerically in, e.g., MATLAB.

## 4. EFFECT OF SAMPLING

The Fisher information matrix for a signal in non-white Gaussian noise is given by [3]

$$J_{ij} = 2\Re\left[\frac{\partial \mathbf{s}(\theta)}{\partial \theta_i}^H \mathbf{R}^{-1} \frac{\partial \mathbf{s}(\theta)}{\partial \theta_j}\right] \quad (13)$$

where in this case $\mathbf{R}$ is the correlation matrix of the noise, $v[k]$, $\mathbf{s} = [s[0;\theta], \ldots, s[N-1;\theta]]^T$, and the derivatives are,

$$\frac{\partial s}{\partial \omega}[k;\theta] = AH'(\omega)e^{i(\omega T_s k + \phi)} + iT_s kAH(\omega)e^{i(\omega T_s k+\phi)} \quad (14)$$

$$\frac{\partial s}{\partial \phi}[k;\theta] = iAH(\omega)e^{i(\omega T_s k + \phi)}. \quad (15)$$

Notice that since $v[k]$ is a stationary random process, $\mathbf{R}$ is a symmetric, Toeplitz matrix and the Levinson-Durbin algorithm can be used for calculating $\mathbf{R}^{-1}$. With oversampling $\mathbf{R}$ is close to singular, and we found that a direct calculation of $\mathbf{R}^{-1}$ in MATLAB was unstable, whereas Levinson-Durbin was stable[2].

The Levinson-Durbin algorithm can also be used for calculating approximate and asymptotic CRB, see [7], where also the MLE is derived.

## 5. OPTIMIZATION OF SAMPLING

There are a number of parameters related to sampling that influence the variance: the order $n$ of the Butterworth filter, the cutoff frequency $\omega_c$ of the Butterworth filter, and the sampling frequency $\omega_s$. Ordinarily, if $\omega_m$ is the maximal signal frequency, these frequencies are chosen heuristically so that $\omega_m < \omega_c < \omega_s/2$. We have instead optimized these frequencies to minimize the variance of the frequency estimate $\hat{\omega}$. In general, the variance may be frequency dependent because of aliasing, and we will therefore minimize

$$\overline{\mathrm{CRB}}[\omega_c, \omega_s] = \max_{\omega} \mathrm{CRB}(\hat{\omega})[\omega, \omega_c, \omega_s].$$

There are two different scenarios to be considered, with different solutions.

First, if the signal is given in a certain time interval $T$, the optimal solution is of course to let the sampling frequency $\omega_s$ tend towards infinity, since aliasing will then totally be eliminated. The number of samples taken during the time interval and the computational complexity will therefore also tend towards infinity, and thus the optimum sampling frequency is a trade off between variance and complexity. Then, for a given sampling frequency, we can optimize the variance with respect to $\omega_c$.

On the other hand, suppose that the number of samples is fixed, typically a power of two considering FFT processing. If the sampling frequency is increased the total time $T$ spanned by the samples is decreased, and as the variance is approximate proportional to $T^{-3}$, this increases the variance considerably. Thus, in this case there will be an optimal set $(\omega_s, \omega_c)$, where we expect $\omega_s$ to be close to $2\omega_m$

In both cases the optimization cannot be performed analytically, and we therefore calculated $\overline{\mathrm{CRB}}(\omega_c, \omega_s)$ for different values of $\omega_c$ and $\omega_s$ and optimized by grid search. The results can be seen in Table 1.

## 6. CORRELATION BASED ESTIMATORS

The autocorrelation sequence (ACS) of $x[k]$ in (2) is given by (for $T_s = 1$ and $H(\omega) = 1$)

$$r_m = Ex[k]x^*[k-m] = A^2 e^{i\omega m} + \sigma^2 \delta_{m,0}. \tag{16}$$

From (16) it is evident that information about the frequency is gathered in the phase of $r_m$ for $m \neq 0$. From data $\{x[0] \ldots x[N-1]\}$ the unbiased estimate of the ACS is

$$\hat{r}_m = \frac{1}{N-m} \sum_{k=m}^{N-1} x[k]x^*[k-m], \quad m = 1, \ldots, N-1 \tag{17}$$

[2]Notice that the standard LEVINSON algorithm in MATLAB does not use the Levinson-Durbin algorithm but LU factorization, and is therefore no more stable than direct calculation of $\mathbf{R}^{-1}$.

Table 1: Optimal sampling and filter frequencies for a Butterworth filter of order 4. In the first part, the sampling frequency $f_s$ and the filter cut-off frequency $f_c$ were simultaneously optimized. In the second part, $f_s$ was fixed at 1.5. In the last part the signal was oversampled 4 times, i.e. the actual number of samples taken was $4N$; $f_c$ was fixed at $0.42$. The CRB is relative to the CRB for the ideal case.

| Optimal sampling | | | | $f_s = 1.5$ | | $f_s = 4.4$ |
|---|---|---|---|---|---|---|
| $N$ | $f_s$ | $f_c$ | CRB | $f_c$ | CRB | CRB |
| 8 | 1 | 0.60 | 1.7 | 0.47 | 2.8 | 0.8 |
| 16 | 1 | 0.60 | 1.9 | 0.47 | 3.1 | 1.0 |
| 32 | 1.12 | 0.48 | 1.9 | 0.47 | 3.2 | 1.2 |
| 64 | 1.10 | 0.43 | 1.8 | 0.47 | 3.3 | 1.2 |
| 128 | 1.10 | 0.39 | 1.7 | 0.50 | 3.3 | 1.3 |

Table 2: Correlation based frequency estimators. $\alpha$ and $\beta_m$-values for differential implementation according to (19).

| | $\alpha$ | $\beta_m$ |
|---|---|---|
| Delta function $V_m = \delta_{m,M}$ | $\frac{1}{M}$ | $1$ |
| Rectangular $V_m = 1$ | $\frac{2}{M(M+1)}$ | $(M+1-m)$ |
| Linear $V_m = m$ | $\frac{3}{M(M+1)(2M+1)}$ | $M(M+1) - m(m-1)$ |

(and $\hat{r}_{-m} = \hat{r}_m^*$). A general frequency estimator based on the sequence $\{\hat{r}_1, \ldots, \hat{r}_M\}$ can be written as weighted average of the phase angles of $\hat{r}_m$, that is

$$\hat{\omega} = \frac{\sum_{m=1}^{M} V_m \angle[\hat{r}_m]}{\sum_{m=1}^{M} V_m m}. \tag{18}$$

where $V_m$ is a weighting function, and $\angle[\cdot]$ denotes the phase angle. The variable $M$ ($1 \leq M \leq N-1$) roughly determines the trade-off between numerical complexity and statistical accuracy. For $M = 1$ the estimator (18) reduces to $\hat{\omega} = \angle[\hat{r}_1]$, an estimator known as the linear predictor frequency estimator, but also known as Pisarenko's harmonic decompositor. In literature, some specific weighting functions have been considered, for example the delta function $V_m = \delta_{m,M}$ [1], uniform $V_m = 1$ and linear $V_m = m$ [5]. For $M > 1$ a direct implementation of (18) has to be combined with some phase unwrapping procedure. Alternatively, the estimator (18) can be rewritten in differential form. Let $\hat{\Phi}(m) = \angle[\hat{r}_m \hat{r}_{m-1}^*]$, $m = 1, \ldots, M$. Then, (18) can be written as

$$\hat{\omega} = \alpha \sum_{m=1}^{M} \beta_m \hat{\Phi}(m). \tag{19}$$

Some $\alpha$ and $\beta_m$ for different weighting functions are listed in Table 2. See [7] for details.

The asymptotic error variance of (18) (or (19)) is a function of $N$ and SNR, but also depends on the window $V_m$ and the number of correlations $M$. The following result holds true.

$$\mathrm{var}[\hat{\omega}] = \frac{1}{S_3(M, V_m)^2} \left( \frac{S_1(M, V_m, N)}{\mathrm{SNR}} + \frac{S_2(M, V_m, N)}{2\mathrm{SNR}^2} \right), \tag{20}$$

where var$[\hat{\omega}]$ denotes the asymptotic error variance, and

$$\begin{aligned} S_1(M, V_m, N) &= \sum_{m=1}^{M}\sum_{n=1}^{M} \frac{V_m V_n \min(m, n, N-m, N-n)}{(N-m)(N-n)}, \\ S_2(M, V_m, N) &= \sum_{m=1}^{M} \frac{V_m^2}{N-m}, \qquad (21) \\ S_3(M, V_m) &= \sum_{m=1}^{M} m V_m. \end{aligned}$$

The proof of (20)-(21) follows the step of the proof in [5] where var$[\hat{\omega}]$ for the special case $V_m = m$ was derived. The details are omitted in the interest of brevity.

For one-bit quantized data it no longer holds that $\hat{r}_m \simeq r_m$. For large $N$ and small SNR it holds, [1]

$$\hat{r}_m \simeq \begin{cases} 2 & m = 0 \\ \frac{4}{\pi}\mathrm{SNR}e^{i\omega m} & m = 1, \ldots, M \end{cases} . \qquad (22)$$

Thus the correlation based estimator provide approximately unbiased estimates for SNR slightly above their SNR-threshold. In the medium SNR region, the error variance is approximately given by (20)-(21) and the transformation SNR $\rightarrow \frac{2}{\pi}$SNR.

For large SNR the estimate no longer will be unbiased. An expression for the asymptotic bias (as SNR $\rightarrow \infty$) is derived next,

$$\begin{aligned} \mathrm{bias}[\hat{\omega}] &= \lim_{\mathrm{SNR}\rightarrow\infty} [\hat{\omega} - \omega] \\ &= \frac{\sum_{m=1}^{M} V_m \lim_{\mathrm{SNR}\rightarrow\infty} \angle[\hat{r}_m r_m^*]}{\sum_{m=1}^{M} V_m m}. \qquad (23) \end{aligned}$$

One can show that, [7]

$$\lim_{\mathrm{SNR}\rightarrow\infty} \angle[\hat{r}_m r_m^*] = \angle\left[\sum_{k=m}^{N-1} z[k] z^*[k-m]\right] \qquad (24)$$

where $z[k] = [e^{i(\omega k+\phi)}]_+ e^{-i(\omega k+\phi)}$ .

## 7. IMPLEMENTATIONAL ASPECTS OF CORRELATION BASED 1 BIT ESTIMATORS

The advantage of using 1 bit sampling and correlation based estimators is that the correlation can be calculated very simply by xor'ing $\mathbf{x} = [x[1], \ldots, x[N-1]]^T$ with a shifted version of itself, and counting the number of ones. The phase angle calculation can be done using table lookup (if 32 1 bit samples with 4 times oversampling is used (in total 256 samples for real and complex part), all values of $\hat{r}_m$ can be represented by 8 bits for the real/complex part, and a table with 64k words is needed). This part of the processing can be done in dedicated hardware. The phase unwrapping and multiplication by $V_m$ can be done in software on a DSP. Further aspects on efficient implementation can be found in [7].

## 8. SIMULATION RESULTS

Figure 2 shows the variance of the frequency estimate for different estimators as a function of the frequency for $N = 32$ samples. For the FFT based estimator we used 4 times zeropadding, and peakfinding by triple Gaussian interpolation, while the correlation

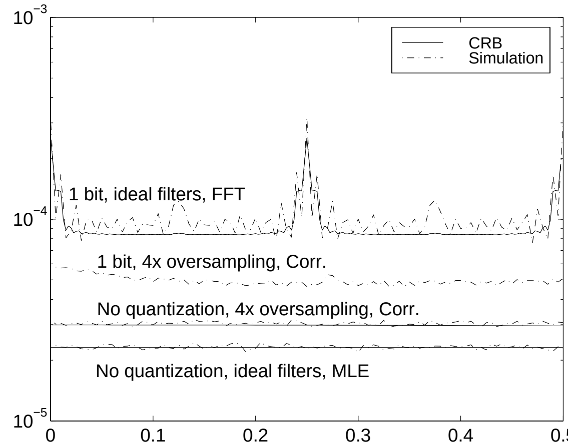


Figure 2: Variance versus frequency for $N = 32$ and SNR=12 dB. The results for 4 times oversampling were obtained with a Butterworth filter of order 4 with a cutoff at 1. The number of ensembles for each frequency was 10000 with random phase.

based estimator used the parabolic window $V_m = m(N-m)$ and $M = N-1$, that is $\alpha = 2/(N^2-1)N^2$ and $\beta_m = (N^2-1)N - (m-1)m(3N-2m+1)$, [6]. It can be seen that there is a good agreement between the CRB and the FFT for 1 bit sampling (which is not the exact MLE for 1 bit sampling). To avoid the variance peaks, the input was then 4 times oversampled. The sampling frequency was 4.4 and the signal was frequency shifted 0.55 prior to sampling in order to make all frequencies between two variance peaks, and a 4th order Butterworth filter was used, with cutoff 1.0. It can be seen that the variance peaks are then completely avoided, and furthermore the general variance level is much reduced and almost reaches the case of no quantization (the fraction is less than $\frac{\pi}{2}$).

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