AN UNBIASED EQUATION-ERROR-BASED ADAPTIVE IIR FILTERING ALGORITHM

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ABSTRACT

We modify the off-line system identification procedure proposed by Regalia [4] into an adaptive IIR filtering algorithm based on the stochastic gradient method. The proposed algorithm aims to minimize equation error, recursively, under a unit-norm constraint on the characteristic polynomial instead of the usual monic constraint. The unit-norm constraint eliminates the bias associated with equation error based estimates, when the additive measurement noise is white. The unit-norm constraint is enforced by adapting the parameters of the characteristic polynomial in (hyper)spherical coordinates. Simulation results indicate that the proposed algorithm provides estimates that are unbiased and that it is a computationally efficient alternative, for the same performance, to FIR adaptive filters.

I. INTRODUCTION

Traditionally, finite impulse response (FIR) structures have been used for adaptive filters, due to their simplicity. However, it could be advantageous to be able to use infinite impulse response (IIR) structures rather than FIR structures for adaptive filters, especially when the desired filter can be modeled with fewer parameters using both poles and zeros rather than using only zeros. The potential reduction in computational complexity and improvement in performance have motivated research in adaptive IIR filtering. A rich repertoire of algorithms for pole-zero modeling, also known as auto-regressive moving-average (ARMA) modeling, is already available in the system identification literature. This has led to the use of certain system identification techniques for adaptive filtering [5]. In this paper, we modify a system identification method proposed by Regalia [4] into an adaptive IIR filtering algorithm based on the stochastic gradient technique.

The system identification method proposed by Regalia [4] uses an off-line procedure, where a batch of data is collected from the system and the collected data is used to construct a model with a separate (off-line) procedure. Off-line procedures are unsuitable for adaptive filtering, where a model of the plant, possibly time varying, is needed during the real time operation of the system. Here, a procedure is desired that updates the model after the arrival of each new data point. Such recursive procedures use less memory than off-line procedures, since there is no need to store all past data, and they may be used as computationally-robust alternatives for the off-line identification methods. Of course we need to address the new issue of convergence, when we use recursive methods.

Section II of this paper discusses the bias in equation error based estimates and a modification to surmount this problem for white measurement noise. An adaptive filtering algorithm based on unbiased equation error minimization is presented in Section III. Results from the simulation of the proposed algorithm are shown in Section IV. Section V provides the conclusion.

II. UNBIASED EQUATION ERROR

The equation error [3] e_n, as shown in Figure 1, is characterized by the difference equation:

$$e_n = \sum_{k=0}^{M} b_k x_{n-k} - \sum_{k=0}^{M} a_k w_{n-k} \ , \tag{1}$$

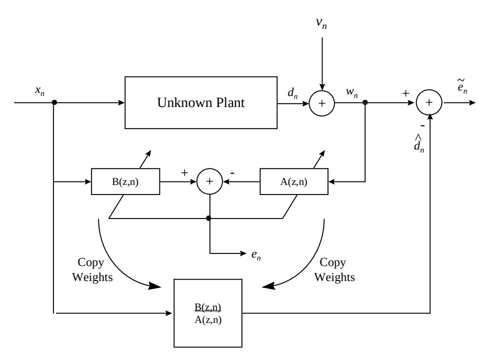

Figure 1. Equation Error Based Identifier used for Adaptive Filtering.

where x_n and w_n are the input and output (corrupted by measurement noise) of the unknown plant, M is the order of the model, and $\{b_k,\ a_k\}$ are the coefficients of the IIR model $H(z)$, that produces the estimate of the plant output, and is defined as

$$\hat{H}(z)=\frac{B(z)}{A(z)}=\frac{\sum_{k=0}^{M} b_k z^{-k}}{\sum_{k=0}^{M} a_k z^{-k}}. \quad (2)$$

The coefficients of the model are chosen so that the mean square equation error

$$E\left[e_n^2\right]=\begin{bmatrix}\mathbf{b}^t & -\mathbf{a}^t\end{bmatrix}\begin{bmatrix}E\left(\mathbf{x}_n\mathbf{x}_n^t\right) & E\left(\mathbf{x}_n\mathbf{w}_n^t\right)\\ E\left(\mathbf{w}_n\mathbf{x}_n^t\right) & E\left(\mathbf{w}_n\mathbf{w}_n^t\right)\end{bmatrix}\begin{bmatrix}\mathbf{b}\\ -\mathbf{a}\end{bmatrix}$$

$$=\begin{bmatrix}\mathbf{b}^t & -\mathbf{a}^t\end{bmatrix}\begin{bmatrix}\mathbf{R}_{xx,n} & \mathbf{R}_{xw,n}\\ \mathbf{R}_{xw,n}^t & \mathbf{R}_{ww,n}\end{bmatrix}\begin{bmatrix}\mathbf{b}\\ -\mathbf{a}\end{bmatrix}, \quad (3)$$

where

$$\mathbf{b}=\begin{bmatrix}b_0 & b_1 & \cdots & b_M\end{bmatrix}^t,$$
$$\mathbf{a}=\begin{bmatrix}a_0 & a_1 & \cdots & a_M\end{bmatrix}^t,$$
$$\mathbf{x}_n=\begin{bmatrix}x_n & x_{n-1} & \cdots & x_{n-M}\end{bmatrix}^t, \text{ and}$$
$$\mathbf{w}_n=\begin{bmatrix}w_n & w_{n-1} & \cdots & w_{n-M}\end{bmatrix}^t,$$

is minimized. If we assume that the signals are stationary and that the measurement noise v_n (in Fig. 1) is a stationary white noise, with variance σ_v^2, which is independent of the input, then (3) may be rewritten as

$$E\left[e_n^2\right]=\begin{bmatrix}\mathbf{b}^t & -\mathbf{a}^t\end{bmatrix}\begin{bmatrix}\mathbf{R}_{xx} & \mathbf{R}_{xd}\\ \mathbf{R}_{xd}^t & \mathbf{R}_{dd}\end{bmatrix}\begin{bmatrix}\mathbf{b}\\ -\mathbf{a}\end{bmatrix}+\sigma_v^2\mathbf{a}^t\mathbf{a}, \quad (4)$$

where d_n is the true output of the plant. A minimizing solution of (4) is $\mathbf{a}=\mathbf{b}=\mathbf{0}$. Some constraint is needed to avoid this trivial solution. Typically, a_0 is set to 1. This results in a monic characteristic polynomial $A(z)$. With the monic constraint the mean square equation error is

$$E\left[e_n^2\right]=\begin{bmatrix}\mathbf{b}^t & -\mathbf{a}^t\end{bmatrix}\begin{bmatrix}\mathbf{R}_{xx} & \mathbf{R}_{xd}\\ \mathbf{R}_{xd}^t & \mathbf{R}_{dd}\end{bmatrix}\begin{bmatrix}\mathbf{b}\\ -\mathbf{a}\end{bmatrix}+\sigma_v^2\left(1+\sum_{k=1}^{M}a_k^2\right). \quad (5)$$

Clearly, the "unwanted" second term on the right hand side of (5) adds a penalty function proportional to the norm of $\mathbf{a}$. This introduces an undesirable bias, which depends on the variance of the measurement noise, to the minimizing solution.

Regalia's identification procedure [4] overcomes this bias problem by using a unit-norm constraint on $\mathbf{a}$ instead of the monic constraint. With this unit-norm constraint, the equation error in (4) becomes

$$E\left[e_n^2\right]=\begin{bmatrix}\mathbf{b}^t & -\mathbf{a}^t\end{bmatrix}\begin{bmatrix}\mathbf{R}_{xx} & \mathbf{R}_{xd}\\ \mathbf{R}_{xd}^t & \mathbf{R}_{dd}\end{bmatrix}\begin{bmatrix}\mathbf{b}\\ -\mathbf{a}\end{bmatrix}+\sigma_v^2. \quad (6)$$

Since σ_v^2 is a constant, minimizing (6) is equivalent to minimizing the equation error under the noise-free condition, i.e., noise does not influence the solution. The Hessian matrix of the mean square equation error, the matrix in the quadratic expression in (6), is positive semidefinite. Hence, the error surface is convex [1]. Furthermore, if the input signal is persistently exciting of degree $2M+1$ and the model order M equals the true order of the unknown plant, the minimizing solution of (6) gives the true parameters of the plant. If M is greater than the true order, the minimizing solution of (6) gives the true parameters after common poles and zeros in the model are canceled [4]. These properties of the unbiased equation error criterion in (6) make it attractive for adaptive filtering using gradient based algorithms.

III. RECURSIVE ADAPTIVE FILTERING ALGORITHM

The objective is to recursively minimize the mean squared value of the equation error shown in (1), under the unit-norm constraint $\mathbf{a}^t\mathbf{a}=1$. The gradient descent algorithm [3], with suitable step-size, can be used to minimize the mean square equation error. However, true gradient computation requires estimation of second order statistics of the signals. This can be avoided by using the stochastic gradient algorithm [3]. Here, the instantaneous squared value of the equation error, rather than its mean, is minimized. We enforce the unit-norm constraint by using a (hyper)spherical parametrization of $\mathbf{a}$. That is, rather than directly adapting $\mathbf{a}=(a_0,a_1,\ldots,a_M)$, we instead adapt $\boldsymbol{\theta}=(\theta_1,\theta_2,\ldots,\theta_M)$, where

$$\begin{aligned} a_0 &= r\cos\theta_1,\\ a_k &= r\left(\prod_{i=1}^{k}\sin\theta_i\right)\cos\theta_{k+1},\ \forall k\in\{1,2,\ldots,M-1\} \quad (7)\\ a_M &= r\prod_{i=1}^{M}\sin\theta_i, \end{aligned}$$

The following constraints are added to have a one-to-one correspondence between $\hat{H}(z)$ of order M and $\{\mathbf{b},\boldsymbol{\theta}\}$:

$$-\pi/2\le\theta_1\le\pi/2, \text{ and}$$
$$-\pi\le\theta_k\le\pi, \forall k\in\{1,2,\ldots,M\}.$$

If $r=1$, the a's, as defined in (7), always satisfy the unit-norm constraint.

The stochastic gradient method based adaptive filtering algorithm is shown in Table 1.

Table 1. Adaptive IIR Filtering Algorithm.

Initialization:

$$\theta_0 = (0, \quad 0, \quad \cdots, \quad 0)^t$$

$$\mathbf{b}_0 = (0, \quad 0, \quad \cdots, \quad 0)^t$$

For $n = 0,1,2,\ldots$, repeat (8) - (11):

$$a_{n,0} = \cos\theta_{n,1} \tag{8 a}$$

$$a_{n,k} = \left(\prod_{i=1}^{k} \sin\theta_{n,i}\right)\cos\theta_{n,k+1}, \ \forall k \in \{1,2,\ldots, M-1\} \tag{8 b}$$

$$a_{n,M} = \prod_{i=1}^{M} \sin\theta_{n,i} \tag{8 c}$$

$$e_n = \sum_{k=0}^{M} b_{n,k} x_{n-k} - \sum_{k=0}^{M} a_{n,k} w_{n-k} \tag{9}$$

$$b_{n+1,k} = b_{n,k} - \mu e_n x_{n-k}, \text{for } k = 0,1,\ldots, M \tag{10}$$

$$\theta_{n+1,k} = \theta_{n,k} - \mu e_n \mathbf{w}_n^t \left(\frac{\partial \mathbf{a}}{\partial \theta_k}\right)_{\theta=\theta_n}, \text{for } k = 1,2,\ldots, M \tag{11}$$

The range of admissible values of μ, to ensure convergence of this algorithm, depends on the eigenvalues of the matrix appearing in (6). For these values of μ, this algorithm converges in mean and mean square sense to a stationary point of the equation error surface, where $\nabla_{\mathbf{b},\boldsymbol{\theta}} E = \mathbf{0}$, in $\mathbf{B} \times \Theta$ space. The Jacobian $\mathbf{J}$ of the transformation (7) relates $\nabla_{\mathbf{b},\boldsymbol{\theta}} E$ to the gradient with respect to $\{\mathbf{b}, \mathbf{a}\}$ parameters as follows:

$$\nabla_{\mathbf{b},\mathbf{a}} E = \begin{pmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{J}^t \end{pmatrix} \nabla_{\mathbf{b},\boldsymbol{\theta}} E \ . \tag{12}$$

Thus, if a point in $\mathbf{B} \times \boldsymbol{\Theta}$ space is a stationary point, then the corresponding point in $\mathbf{B} \times \mathbf{A}$ space is also a stationary point. Since the equation error surface is convex in $\mathbf{B} \times \mathbf{A}$ space, every stationary point here is a global minimum of the equation error. Hence, the above stochastic gradient based algorithm converges (in mean and mean-square sense) to the global minimum of the equation error surface, even though the adaptation is carried out in terms of θ parameters rather than a parameters.

IV. SIMULATION RESULTS

The proposed algorithm is simulated in Matlab. The plant to be modeled is a fifth order system with poles and zeros as shown in Figure 2. The model order M is chosen to be 5, equal to the true system order. The input x_n and the measurement noise v_n are assumed to be white. The signal-to-noise ratio (SNR) of the measured output is 10 dB. The steady-state pole and zero estimates from the proposed algorithm, along with the estimates from the traditional equation error based algorithm (with monic constraint), are shown in Figure 2.

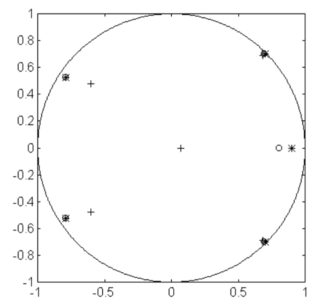

Figure 2(a). True Pole Locations (*), Estimate using Unbiased Equation Error (o), and Estimate using Traditional Equation Error with Monic Constraint(+).

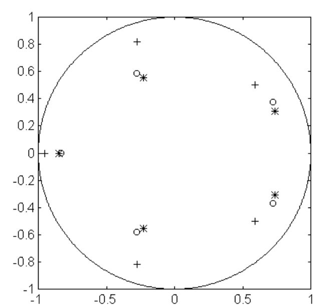

Figure 2(b). True Zero Locations (*), Estimate using Unbiased Equation Error (o), and Estimate using Traditional Equation Error with Monic Constraint(+).

The unbiased equation error approach produces estimated poles that nearly coincide with the original poles, and estimated zeros that are mostly very close to the original zeros. The corresponding results from the monic equation error approach exhibit a pronounced bias. Alternatively we can look at the corresponding magnitude responses shown in Figure 3. Clearly, the estimate from the traditional algorithm is biased, while the estimate from the proposed algorithm is unbiased.

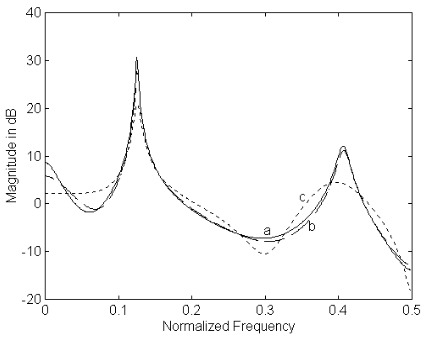

Figure 3. Magnitude Responses: (a) True Response, (b) Estimate from Equation Error with Unit-Norm Constraint, and (c) Estimate from Equation Error with Monic Constraint.

Figure 4 shows the convergence behavior of the estimate for $a_{n,0}$. The estimates for other $a_{n,k}$'s behave similarly.

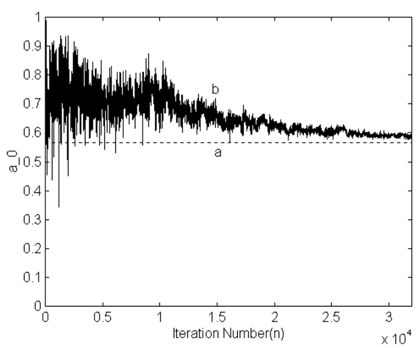

Figure 4. (a) True Value of a_0 and (b) Estimated a_0 from the Adaptive IIR Filtering Algorithm.

The next experiment illustrates the reduction in computational complexity that can result from using an IIR structure instead of an FIR structure. The unknown plant used here is the same 5^{th} order plant discussed earlier. Using a 5^{th} order pole-zero model, we achieve a ratio of estimation error (defined to be $d_n - d_n$) variance to desired signal variance (in the echo-cancellation literature [2], this quantity is usually referred to as Echo Return Loss) of better than -200 dB, as shown in Figure 5. The proposed IIR algorithm requires approximately 50 arithmetic operations per iteration, assuming that the sine and cosine values can be looked up from a table. However, even with a 1024^{th} order FIR model, and using the NLMS algorithm [2] to adapt this FIR model, the ratio of estimation error variance to desired signal variance is not even -100 dB. The NLMS algorithm requires approximately 3000 arithmetic operations per iteration. Consequently, for this example, the FIR structure requires much more computational effort than the IIR structure, and performs considerably less well.

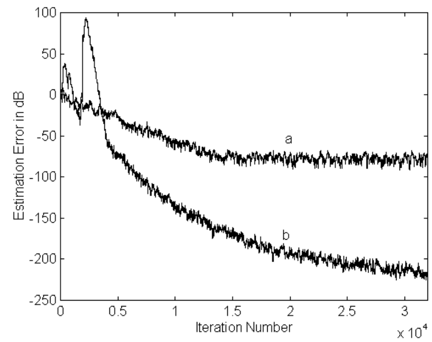

Figure 5. (a) Estimation Error from NLMS(1024) and (b) Estimation Error from the Proposed Algorithm.

A drawback of any IIR approach is that the identified model can be unstable. This manifests itself in Figure 5 by iterations where, temporarily, the estimation error for the proposed algorithm is larger than for the NLMS algorithm. Monitoring model stability can mitigate this problem, by skipping copying of the weights, while eventually yielding the improved performance.

V. CONCLUSION

An algorithm for adaptive IIR filtering is presented. This algorithm does not have any bias problem, if the measurement noise is white, and it converges to the global minimum of the unbiased equation error surface. The transient behavior of this algorithm needs further study. The closer the poles of the unknown plant are to the unit circle, the more the proposed algorithm realizes a computational advantage over the NLMS-based FIR adaptive filtering algorithm.

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