1-D CONTINUOUS NON-MINIMUM PHASE RETRIEVAL USING THE WAVELET TRANSFORM

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ABSTRACT

The phase retrieval problem arises when a signal must be reconstructed from only the magnitude of its Fourier transform; if the phase information were also available, the signal could simply be synthesized using the inverse Fourier transform. In continuous phase retrieval, most previous solutions rely on discretizing the problem and then employing an iterative algorithm. We avoid this approximation by using wavelet expansions to transform this uncountably infinite problem into a linear system of equations. The wavelet bases permit a solution by incorporating a priori signal information and they provide a structured system of equations which results in a fast algorithm. Our solutions obviate the stagnation problems associated with iterative algorithms, they are computationally simpler and more stable than previous non-iterative algorithms, and they can accommodate noisy Fourier magnitude information. This paper develops our 1-D continuous, non-minimum phase retrieval algorithm and illustrates its effectiveness with numerical examples.

1. INTRODUCTION

Reconstructing a 1-D signal with compact support from the modulus of its Fourier transform is known as the phase retrieval problem because if we also had the phase information, we would simply synthesize the signal using the inverse Fourier transform. This problem is associated with various applications including antenna design, filter design, and the characterization of astronomical objects—see [6] for references.

The squared Fourier modulus corresponds to the Fourier transform of the autocorrelation function of the signal $x(t)$. Thus, the 1-D phase retrieval problem can be described by

$$r(t) = \int x(u)x(u-t)\,du; \quad -T \le t \le T$$

where the autocorrelation function $r(t)$ is known. The assumption that the signal has compact support permits its Fourier transform to be sampled in frequency. Furthermore, if the signal is assumed to be bandlimited, it can be sampled in time as well. This corresponds to the 1-D *discrete* phase retrieval problem in which the Fourier transform is replaced by the discrete Fourier transform (DFT). Conventional numerical solutions of the 1-D continuous phase retrieval problem amount to discretizing $r(t)$ and employing iterative phase retrieval algorithms to reconstruct a discrete time signal. However no signal can be both time and bandlimited. Moreover, these iterative algorithms exhibit stagnation (i.e. convergence) problems (see [4] for the original algorithm and refer to [2] for its more recent descendants). Therefore other solutions are sought.

We present a new approach to solving the 1-D continuous phase retrieval problem which avoids these approximations and problems. Our algorithms utilize the wavelet representation of a continuous time signal in order to transform the problem into a discrete, linear system of equations. The advantages of this approach include: a more accurate representation of the continuous time signal; the incorporation of a priori signal information which permits a solution; a structured system of equations which permits a fast algorithm; elimination of the convergence problems associated with iterative phase retrieval algorithms; and, the accommodation of noisy Fourier magnitude information.

In this paper, Section 2 presents a review of the continuous time phase retrieval problem, our minimum phase solution, and the wavelet expansion of a 1-D signal. The non-minimum phase solution is derived in Section 3; its relationship to the minimum phase solution via an underlying differential equation is developed. Section 4 presents several illustrations of our algorithm and Section 5 concludes with the contributions of our work.

2. REVIEW: MINIMUM PHASE RETRIEVAL AND THE WAVELET TRANSFORM

2.1. The 1-D Phase Retrieval Problem

The 1-D phase retrieval problem can be described as follows. Let $x(t)$ be a continuous time signal. Given that $x(t)$ is nonzero only on the interval $[0, T]$ and given the magnitude of its Fourier transform $|X(\Omega)| = |\mathcal{F}x(t)|$, reconstruct the signal $x(t)$.

Since we only have the modulus information, $M(\Omega) = |X(\Omega)|$, we consider $M^2(\Omega) = |X(\Omega)|^2$ with the analytic extension

$$M^2(s) = X(s)X^*(-s^*).$$

$M^2(s)$ is an entire function of exponential type which implies that it is completely specified by its complex zeros [5]. Moreover, the inverse Fourier transform of $M^2(\Omega)$ is the autocorrelation function $r(t)$.

As in discrete phase retrieval, there are both trivial and non-trivial ambiguities in this 1-D, continuous time phase retrieval problem. The trivial ambiguities include constant scale factors and translations. However, the problem is ambiguous beyond these trivial factors because the complex zeros of $M^2(s)$ occur in negative complex conjugate pairs (i.e., the zeros $\{s\}$ of $X(s)$ and the zeros $\{-s^*\}$ of $X^*(-s^*)$). For instance, if $M^2(s)$ has N complex conjugate pairs of zeros, then there are 2^N non-trivial solutions (i.e. 2 ways of choosing one zero from each of the N conjugate pairs). Moreover, of all these non-trivial solutions, there is only one minimum phase solution—that with all its zeros in the open left half plane. We will use this minimum phase solution (along with a priori signal information) to determine the desired non-minimum phase signal.

2.2. Our Minimum Phase Retrieval Solution

In [1], we showed that given $|X(\Omega)|$, we can find the corresponding minimum phase signal $x(t)$. This problem was shown to correspond to the Krein integral equation,

$$\delta(t-s) = x(t-s) + \int_{-t}^{t} x(t-u)k(|u-s|)\,du, \quad 0 \le |s| \le t \tag{1}$$

where k is known and x is the unknown, minimum phase (i.e. causal) signal with compact support. Thus, solving the Krein integral equation for $x(t)$ is equivalent to solving the 1-D minimum phase retrieval problem.

2.3. Discrete Orthonormal Wavelet Transform

The discrete orthonormal wavelet transform $X(m,n)$ of a continuous square-integrable function $x(t)$ is

$$X(m,n) = \int_{-\infty}^{\infty} x(t)2^{m/2}\psi(2^m t - n)\,dt \tag{2}$$

$$x(t) = \sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty} X(m,n)2^{m/2}\psi(2^m t - n) \tag{3}$$

where $\psi(t)$ is the wavelet basis function [3]. $\psi(t)$ is orthogonal (in the sense of the usual L^2 inner product) to its scalings $\psi(2^m t)$ (dilations for $m<0$; compressions for $m>0$) and to the translations $\psi(2^m t - n)$ of its scalings. The set of all scalings and translations $\{2^{m/2}\psi(2^m t - n); m,n \in \mathcal{Z}\}$ forms a complete orthonormal set.

3. OUR NON-MINIMUM PHASE RETRIEVAL SOLUTION

3.1. Relationship to the Minimum Phase Solution

Given this minimum phase signal, what additional information about the desired non-minimum phase signal do we need in order to reconstruct it? $X(s)$, the unique analytic extension of $X(\Omega)$ to the complex s-plane, can be completely characterized by its complex zeros [5]. If we order the zeros of $X(s)$ such that the first k are in the right half plane (RHP) and the remainder are in the open left half plane (OLHP), then we can order the zeros of $X_{MIN}(s)$ such that its first k zeros correspond to those k RHP zeros of $X(s)$ flipped into the OLHP. The remaining $X_{MIN}(s)$ zeros are the same as the remaining $X(s)$ zeros. The zeros are flipped by $X_{FLIP}(s) = \prod_{i=1}^{k}(s-s_i)$ where $s_i < 0$ means that all the zeros of $X_{FLIP}(s)$ are in the OLHP. Thus, $X(s)$ and $X_{MIN}(s)$ are related—to within a constant—by

$$X(s) = X_{MIN}(s)\frac{(-1)^k X_{FLIP}(-s)}{X_{FLIP}(s)} \tag{4}$$

where $(-1)^k X_{FLIP}(-s) = \prod_{i=1}^{k}(s+s_i)$ are the zeros of $X_{FLIP}(s)$ flipped into the RHP.

Cross-multiplying and denoting the coefficients of X_{FLIP} by $(-1)^{k-l}x_F^l$ gives

$$\left[\sum_{l=0}^{k}(-1)^{k-l}x_F^l s^l\right] X(s) = \left[\sum_{l=0}^{k} x_F^l s^l\right] X_{MIN}(s). \tag{5}$$

In the time domain, this corresponds to the differential equation (assuming initial conditions are zero)

$$\sum_{l=0}^{k}(-1)^{k-l}x_F^l\frac{d^l x(t)}{dt^l} = \sum_{l=0}^{k} x_F^l\frac{d^l x_{MIN}(t)}{dt^l}. \tag{6}$$

3.2. Linear System of Equations

We use the result of the previous section along with the discrete wavelet transform of the derivative of a signal to arrive at the desired linear system of equations. First we take the discrete wavelet transform of both sides of (6)

$$\sum_{l=0}^{k}(-1)^{k-l}x_F^l DWT\left\{\frac{d^l x(t)}{dt^l}\right\} = \sum_{l=0}^{k} x_F^l DWT\left\{\frac{d^l x_{MIN}(t)}{dt^l}\right\}.$$

Furthermore, we define

$$\Psi^{(l)}(i,j,m,n) = (-1)^l 2^{\frac{((2l+1)m+i)}{2}}\int \psi(2^i t - j)\psi^{(l)}(2^m t - n)\,dt, \tag{7}$$

to arrive at our desired result

$$\sum_{l=0}^{k}(-1)^{k-l}x_F^l\sum_{i=-\infty}^{\infty}\sum_{j=-\infty}^{\infty} X(i,j)\Psi^{(l)}(i,j,m,n) =$$

$$\sum_{l=0}^{k} x_F^l\sum_{i=-\infty}^{\infty}\sum_{j=-\infty}^{\infty} X_{MIN}(i,j)\Psi^{(l)}(i,j,m,n). \tag{8}$$

In equation (8), $X_{MIN}(i,j)$ and $\Psi^{(l)}(i,j,m,n)$ can be calculated while x_F^l and $X(i,j)$ are unknown. However, if the left-hand side is zero for some scales and translations, the flip coefficients x_F^l can be found from the null space of a known matrix on the right-hand side. Then our desired non-minimum phase signal may be found from equation (4).

3.3. Zeros on the Left-hand Side

Smooth regions of a signal map to zero wavelet coefficients at fine enough scales. Of course $X(i,j)=0, \forall(i,j)$ is not interesting, but the idea is to expand $x(t)$ such that $X(i,j)$ has bands of zeros in it for some (i,j). Then equation (8) is zero on the left-hand side and known on the right-hand side for these scales and translations. The flip coefficients can then be determined from this known submatrix. To understand this, consider the following example.

Let $x(t)$ be a signal with compact support on $[0,1]$ and let $x_{MIN}(t)$ be its corresponding minimum phase signal (also with compact support on $[0,1]$). Furthermore, assume we know that $x(t)$ is constant on $[0,\frac{1}{4}]$.

The Haar wavelet expansions of $x(t)$ and $x_{MIN}(t)$ have the following structures

$$X = \begin{bmatrix} 0 & * & * & * & 0 & 0 & 0 & 0 \\ 0 & * & * & * & 0 & 0 & 0 & 0 \\ 0 & 0 & * & * & * & * & * & * \end{bmatrix}$$

$$X_{MIN} = \begin{bmatrix} * & * & * & * & 0 & 0 & 0 & 0 \\ * & * & * & * & 0 & 0 & 0 & 0 \\ * & * & * & * & * & * & * & * \end{bmatrix} \quad (9)$$

where the first row corresponds to the scaling function at scale 2 ($\phi(4t)$), the second row corresponds to the wavelet function at scale 2 ($\psi(4t)$), and the third row corresponds to the wavelet function at scale 3 ($\psi(8t)$).

At scale $m=3$, the Haar basis function has width $\frac{1}{2^3} = \frac{1}{8}$ and the translations $n=0$ and $n=1$ correspond to the two intervals $[0,\frac{1}{8}]$ and $[\frac{1}{8},\frac{1}{4}]$. Thus, for $k=1$, the basis functions of Ψ are completely within the constant region of $x(t)$ at $m=3; n=0,1$. This means that $X * \Psi^1_{3,0} = 0$, $X * \Psi^0_{3,0} = 0$, $X * \Psi^1_{3,1} = 0$, and $X * \Psi^0_{3,1} = 0$. However, $X_{MIN} * \Psi^1_{m,n} \neq 0$ and $X_{MIN} * \Psi^0_{m,n} \neq 0$ for $m=3; n=0,1$. Thus, we have reduced equation (8) to a known system for these particular (m,n). As a result, we have a known $2x2$ system matrix whose null space provides the coefficients of $X_{FLIP}(s)$ (10).

$$\begin{bmatrix} X_{MIN} * \Psi^1_{3,0} & X_{MIN} * \Psi^0_{3,0} \\ X_{MIN} * \Psi^1_{3,1} & X_{MIN} * \Psi^0_{3,1} \end{bmatrix} \begin{bmatrix} x^1_F \\ x^0_F \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \quad (10)$$

Notice that the width of a signal's smooth region as well as the width of the chosen wavelet basis function dictate the formulation of the system matrix.

3.4. Procedure

The following procedure outlines how to obtain the non-minimum phase solution $x(t)$ of the 1-D continuous phase retrieval problem.

1. Given $|X(\Omega)|$ and the support of $x(t)$, compute the minimum phase solution $x_{MIN}(t)$ using the algorithm in [1];
2. Given the additional information of the number of zeros to flip, k, and that part of $x(t)'s$ support which is smooth, calculate Ψ using (7) and calculate the wavelet transform of $x_{MIN}(t)$ using (2);
3. At the appropriate scales and translations, formulate the submatrix of the overall system matrix whose null space corresponds to the coefficients of $X_{FLIP}(s)$; and,
4. Use (4) to reconstruct $X(s)$ (which is equivalent to reconstructing $x(t)$).

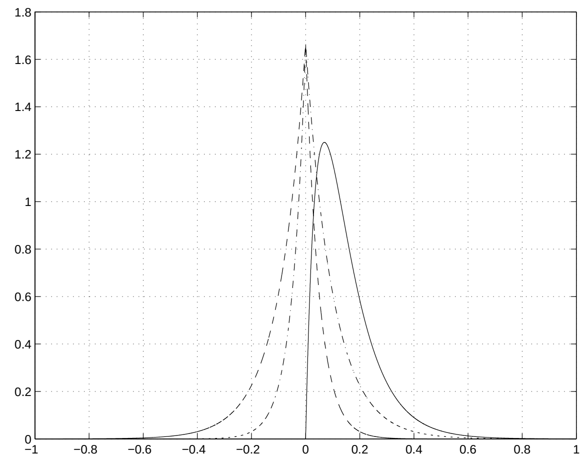

Figure 1: Example 1: $x_{MIN}(t)$–solid, $x(t)$–dashed.

4. EXAMPLES

In determining which wavelet bases would be 'best' in solving this 1-D, continuous, non-minimum phase retrieval problem we are primarily concerned with the regularity and support size properties of the wavelets. In these examples we compare the Daubechies, $N=7$, wavelet basis with the cubic spline Battle-Lemarie wavelet basis [3].

The Daubechies $N=7$ wavelet basis (DAUB7) has compact support on $[0,13]$ and is at least twice differentiable. The cubic spline Battle-Lemarie wavelet basis (BL3) has infinite support and may be differentiated three times. These two bases were chosen because they represent the trade-offs involved in differentiation and compact support: the numerical differentiation for DAUB7 introduces computational errors not seen in BL3 and the truncation of BL3 introduces an approximation error not seen in DAUB7.

The first example we consider is for $x(t) = \frac{5}{3}[e^{10t}u(-t) + e^{-20t}u(t)]$. Its corresponding minimum phase signal is $x_{MIN}(t) = 5[e^{-10t} - e^{-20t}]u(t)$. $X_{MIN}(s)$ has two poles in the complex plane at $s=-10,-20$ and by flipping either one or both of these poles, three non-minimum phase signals can be generated.

Notice that this is not a compact support signal! $x_{MIN}(t)$ has infinite support on the positive real line and $x(t)$ has infinite support on the entire real line. However, both signals are essentially zero for $|t| > 1$ so that we treat them as compact signals.

Figure 1 plots these signals. $x(t)$ is the dashed line, $x_{MIN}(t)$ is the solid line, and the other non-minimum phase signal corresponding to one flipped zero is the dash-dot line. Notice that $x(t)$ is essentially zero for $[0.3,1]$; this is the 'smooth' region of our signal. The other non-minimum phase signal with one flipped zero is not zero or smooth on this interval.

So we begin the reconstruction. We use the given information of $|X(\Omega)|$ and compact support $[-1,1]$ to generate $x_{MIN}(t)$. Then we use this computed $x_{MIN}(t)$ along with the given information of $k=1$ and the subinterval $[0.3,1]$ corresponding to a smooth region in $x(t)$ to generate the system matrix whose null space gives the coefficients of

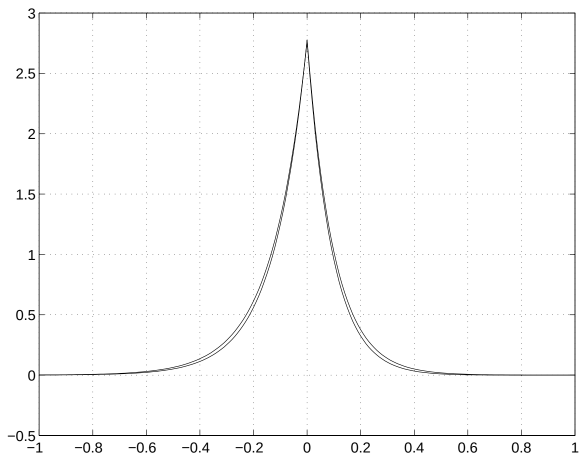

Figure 2: Example 3: actual and reconstructed $x(t)$.

$X_{FLIP}(s)$. Finally, $X(s)$ is computed using $X_{MIN}(s)$ and $X_{FLIP}(s)$ and $x(t)$ can be recovered from $X(s)$.

The actual flip coefficients are $\{1, -10\}$ and the closest computed flip coefficients using DAUB7 were $\{1, -9.9975\}$ while the best approximation using BL3 gave $\{1, -9.9976\}$. The reconstructed $x(t)$ using the computed flip coefficients of $\{1, -9.9975\}$ depicted no discernible difference between the true and reconstructed signals.

The second example we considered used the same $x_{MIN}(t)$, but with $k = 2$. Here we are trying to find the maximum phase signal $x_{MAX}(t)$ which has a smooth region on the interval $[0, 1]$ (it is all zero here). Again, the DAUB7 and BL3 bases performed similarly computing flip coefficients $\{1, -30.0658, 200.6724\}$ compared to the actual flip coefficients of $\{1, -30, 200\}$.

The third and final example we considered was similar to the first in that $k = 1$ and $X_{MIN}(s)$ has two poles; however, now the poles are closer together at $s = -8, -10$.

Now $x(t)$ is essentially zero for $[0.6, 1]$; this is the 'smooth' region of our signal. However the other non-minimum phase signal with one flipped zero is also very close to zero on this interval. So we expect that the reconstruction of this $x(t)$ will not be as accurate as in the first example when the poles were farther apart at $s = -10, -20$.

The actual flip coefficients are $\{1, -8\}$ and the closest computed flip coefficients using DAUB7 were $\{1, -7.5283\}$ while the best approximation using BL3 gave $\{1, -7.5323\}$. We see that DAUB7 and BL3 generated approximately the same answers again; however, the computed flip coefficients are not nearly as accurate here as in the first example (an error of 6.25% compared to 0.025%).

The reconstructed $x(t)$ using the computed flip coefficients of $\{1, -7.5283\}$ is shown in Figure 2. Notice that despite the inaccuracy of the flip coefficients, it is difficult to see a difference between the reconstructed signal and the true signal.

In conclusion, reconstruction of the 1-D, continuous, non-minimum phase signal worked well in these examples. The DAUB7 and BL3 bases reconstructed the desired $x(t)$ similarly well; however, the BL3 had an advantage in that its support size after compression was larger than that for DAUB7. Moreover, the BL3 basis required significantly more computation and storage than the DAUB7 basis due to its large support size ($\phi_{BL3}(t)$ is about 3.5 times as large as $\phi_{DAUB7}(t)$). Thus, the compact support of the wavelet basis appears to be more critical to good performance than whether the differentiation is determined analytically or numerically.

5. CONCLUSION

The results of our work in continuous time phase retrieval are algorithms which make the following contributions.

1. These algorithms permit a better representation of the continuous time signal by using basis functions instead of simply discretizing the signal (i.e. there is now a choice of basis functions).
2. By using wavelet bases, our algorithms incorporate a priori signal information which permits a solution, they represent the problem as a structured system of equations (thus permitting a fast algorithm), and they represent self-similar or smooth signals with few coefficients.
3. These solutions are not iterative; thus, they avoid the convergence problems associated with previous iterative phase retrieval algorithms.
4. These algorithms formulate the problem as a system of linear equations whose solution is computationally simpler and more stable than previous algorithms which rely on the analytic properties of the signal.
5. These algorithms can accommodate noisy Fourier magnitude information better than previous algorithms (via total least squares techniques).

6. REFERENCES

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