STABILIZATION OF STATIONARY AND TIME-VARYING AUTOREGRESSIVE MODELS

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ABSTRACT

A method for the stabilization of stationary and time-varying autoregressive models is presented. The method is based on the hyperstability constrained LS-problem with nonlinear constraints. The problems are solved iteratively with Gauss-Newton type algorithm that sequentially linearizes the constraints. The proposed method is applied to simulated data in the stationary case and to real EEG data in the time-varying case.

1. INTRODUCTION

Stability is often an important issue in such applications as spectral estimation, simulation of time series and decoding of linear prediction coded (LPC) samples. In LPC an AR(p) model is formed for a sample that is then represented by the model parameters and the (possibly quantized) prediction errors. When phase information is not needed, the original signal is simulated simply by feeding white noise to the corresponding filter. If the process is of narrow band type, that is, at least one of the roots has near unit modulus, the roots that correspond to the estimated parameters may be even nearer to the unit circle. In such cases the reconstruction of the process can turn out to be an unstable problem even when the AR parameter estimation scheme guarantees stability (e.g lattice methods and the Yule-Walker method) [9].

In the analysis of EEG signals there are similar difficulties, especially in the case of time-varying AR models [6]. Also in these applications the narrow-bandedness of the process can cause meaningless estimates. These difficulties are related to the lack of Parseval's identity for AR models and to the rise times of very narrow band processes [9].

A further class of problems is related to time-varying AR modeling in which the time-varying parameters are linear combinations of some predetermined basis functions. This scheme is called the time-varying autoregressive least squares method (TVARLS) when the 2-norm of the prediction error is minimized. In this case the estimated models become very easily temporarily unstable thus effectively preventing the end use of the models [4, 7]. In TVARLS modeling the possibility of temporarily unstable models is especially great when the signal contains transitions between narrow-band and wide-band epochs. Such transitions occur e.g. in the modeling of EEG signals [6]. It can be said that this unstability problem has been one of the main hindrances in the applications of TVARLS models. Specifically, it has been suggested that *"It may be possible to develop a time-varying estimation method or to determine sets of basis functions for time-varying LPC that will necessarily lead to stable filters."* [4]. This problem is also stated in [1, 7]. Since it can easily be shown that no set of basis functions guarantees the global stability of a TVARLS model, other approaches have to be pursued.

In this paper we give a method for the estimation of hyperstable AR models. A stationary AR(p) model is stable, if the roots $\lambda_k,\ k = 1, \ldots, p$ of the associated characteristic polynomial have modulus less than unity. Correspondingly, a time-varying AR(p) model is stable, if the above holds for the roots $\lambda_k(t),\ k = 1, \ldots, p$ at each time t. We discuss here only the nonwindowed LS method. The rest of the paper is organized as follows. In Section 2 we formulate the hyperstability constrained AR parameter estimation problem and present an algorithm for its solution in the stationary case. In Section 3 we extented the stability constraints to the time-varying case. In Section 4 we study examples that illustrate the performance of the algorithm.

2. HYPERSTABILITY CONSTRAINED AR ESTIMATION, STATIONARY CASE

The hyperstability constrained least squares AR parameter estimation problem is of the form

$$\min_a \|Ha - X\|_2 \ , \ |\lambda_k(a)| \le \rho \ , \ \forall \ k = 1, \cdots, p \ , \quad (1)$$

where $\rho < 1$, $\lambda_k(a)$ are the roots of the characteristic polynomial, $a = (a_1, \cdots, a_p)$ are p^{th} order predictor coefficients, $X = (x_{p+1}, \cdots, x_T)^{\text{T}}$ and $H = (H_1, \cdots, H_p)$, $H_j = (x_{p+1-j}, \cdots, x_{T-j})$ and $(\cdot)^{\text{T}}$ refers to matrix transposition. The constraints are nonlinear inequality constraints for which there is no explicit form. This constitutes the main difficulty of the problem. We solve the problem iteratively using a Gauss-Newton type algorithm that sequentially linearizes the constraints.

It can be shown by using the perturbation theoretic approach of the characteristic polynomial's companion matrix that problem (1) can be tranformed to least squares problem with inequality constraints [5]

$$\min_{\delta a} \|H\delta a - (X - Ha)\|_2 \ , \quad F(a_0) + J_F(a_0)\delta a \le 0 \quad (2)$$

where δa is the change in a corresponding to the perturbation of the companion matrix, $F(a_0)$ is the constrait function

$$F(a) = \left(|\lambda_1(a)|^2 - \rho^2, \cdots, |\lambda_p(a)|^2 - \rho^2 \right) \quad (3)$$

and $J_F(a_0)$ is the Jacobian of F. It can be also shown [5] that the elements of the Jacobian are obtained as

$$(J_F\,(a_0))_{k,j} = 2\left(\lambda_k^{\text{r}}\alpha_{k,j}^{\text{r}} + \lambda_k^{\text{i}}\alpha_{k,j}^{\text{i}}\right) \ , \quad (4)$$

where $\alpha_{k,j}^{\text{r}}$ and $\alpha_{k,j}^{\text{i}}$ are the j^{th} components of the vector $\alpha_k^{\text{r}} = \Re\{\left(w_k^{\text{H}}v_k\right)^{-1} w_{k,1}^{\text{H}} v_k^{\text{T}}\}$ and $\alpha_k^{\text{i}} = \Im\{\left(w_k^{\text{H}}v_k\right)^{-1} w_{k,1}^{\text{H}} v_k^{\text{T}}\}$, respectively, where $(\cdot)^{\text{H}}$ denotes the complex conjugate transpose, $\Re$ the real part and $\Im$ the imaginary part respectively, w_k^{H} is the left eigenvector of the companion matrix, v_k is the corresponding right eigenvector and $w_{k,1}^{\text{H}}$ is the first element of w_k^{H}. The problem (2) is a least problem with inequality constraints which can be solved with a method presented in [8].

The solution to the original problem (1) is sought iteratively by

$$a^{(\ell)} = a^{(\ell-1)} + \mu\delta a^{(\ell)} \quad (5)$$

where ℓ denotes the iteration number, μ is a step size and in (2) we replace a with $a^{(\ell)}$, a_0 with $a^{(\ell-1)}$ and δa with $\mu\delta a^{(\ell)}$. If a_* is a local minimizer of the original problem (1), then by the Kuhn-Tucker conditions a_* is also the solution of (2). Since the solutions to the linearized problems are not necessarily in the feasible region, each $a^{(\ell)}$ has to be projected onto the feasible region before the problem is linearized. The projection can be done for example by calculating the roots corresponding to $a^{(\ell-1)}$ and adjusting their moduli if necessary.

The termination of the iteration can be done by observing the angle between the gradient of the objective functional and the hyperplane that is normal to the active (linearized) constraint set. The active set refers to those constraints in (2) for which equality holds. The details of the proposed method can be found in [5].

3. THE TIME-VARYING EXTENSION

In the basis constrained time-varying AR problem (TVAR) the coefficients are restricted as

$$a_k(t) = \sum_{\ell=0}^{M} c_{k\ell}\phi_\ell(t) \ , \quad (6)$$

where $\phi_\ell(t), \ \ell = 0, \ldots, M$ are the basis functions. The basis constrained LS problem can be stated as

$$\min_C \|X - HC\|_2 \ , \quad (7)$$

the solution of which is

$$C = (H^{\text{T}}H)^{-1}H^{\text{T}}X \ . \quad (8)$$

where $C = (c_{10}, \ldots, c_{1M}, \ldots, c_{p0}, \ldots, c_{pM})^{\text{T}}$ and regressor matrix $H = (H_{10}, \ldots, H_{1M}, \ldots, H_{p0}, \ldots, H_{pM})$ where $H_{k\ell} = (\phi_\ell(p+1)x_k, \ldots, \phi_\ell(T)x_{T-p+k})^{\text{T}}$ The time-varying coefficients are then assembled via (6).

The hyperstability constrained estimation scheme is extended to the time-varying case as follows. In this case we demand that the roots of the (frozen time) characteristic polynomial have to lie inside the ρ-circle in complex plane for every time instant. Thus the hyperstability constrained TVARLS problem is of the form

$$\min_C \|HC - X\|_2 \ , \ |\tilde{\Lambda}(t)| \le \rho \ , \ t = p+1, \ldots, T \quad (9)$$

where $\tilde{\Lambda}(t) = (\lambda_1(a(t)), \ldots, \lambda_p(a(t)))^{\text{T}}, \ k = 1, \ldots, p$ and $a(t)$ is a function C as in (6). Also in this case the problem can be solved with algorithm presented in 2 by linearizing the constraints. For that we use the same basis function set for time dependent change in $a_k(t)$ *i.e.*

$$\delta a_k(t) = \sum_{\ell}^{M} \delta c_{k\ell}\phi_\ell(t) \ , \quad k = 1, \ldots, p \quad (10)$$

or in matrix form

$$\delta a(t) = \Phi(t)\delta C \quad (11)$$

where $\delta a(t) = (\delta a_1(t), \cdots, \delta a_p(t))^{\text{T}}$, δC is the vector corresponding to an adjustment of C and $\Phi(t)$

is a block diagonal matrix whose diagonal blocks are $(\phi_1(t), \cdots, \phi_M(t))$. Now replacing δa with $\delta a(t)$ in (2) and using δC as the independent variable we obtain

$$\min_{\delta C} \|H\delta C - (X - HC_0)\|_2 \ , \ \tilde{F} + \tilde{J}\delta C \leq 0 \qquad (12)$$

where C_0 is the linearization center, $\tilde{F} = (F(a_0(p+1)), \cdots, F(a_0(T)))^{\mathrm{T}}$ and $\tilde{J} = (J_F(a_0(p+1))\Phi(p+1), \cdots, J_F(a_0(T))\Phi(T))^{\mathrm{T}}$. We have thus obtained a TVARLS problem with linear inequality constraints. The iteration corresponding to (12) is somewhat more cumbersome than in the stationary case since the set of active constraints can vary considerably during the iteration.

4. EXPERIMENTAL RESULTS

4.1. The stationary case

We study the constrained parameter estimation problem of a 4^{th} order AR process. The sample is of length $T = 256$ and the matrix H corresponds to the prewindowed forward prediction equations. The roots of the characteristic polynomial of the process are

$$\lambda_{1,2} = 0.92\exp(\pm 0.20\pi\mathtt{i}) \ , \quad \lambda_{3,4} = 0.60\exp(\pm 0.32\pi\mathtt{i})$$

We choose $\rho = 0.9$. The unconstrained least squares estimate gives the prediction coefficients

$$a = (-2.1225,\ 2.1546,\ -1.0797,\ 0.3103)\ .$$

and the corresponding roots

$$\begin{aligned} \lambda_{1,2} &= 0.9223\exp(\pm 0.1996\mathtt{i}) \\ \lambda_{3,4} &= 0.6039\exp(\pm 0.3258\mathtt{i}) \end{aligned}$$

As the initial value of the iteration we use the polynomial coefficients that are obtained by moving the roots that correspond to the unconstrained LS solution radially inside the feasible region.

The step size $\mu = 0.2$ was chosen and the algorithm iterates until the inner products of the gradient and the active linearized constraints are between $[-\epsilon, \epsilon]$. We used the tolerance $\epsilon = 10^{-4}$. Table 1 shows the angles ψ between the active constraint planes and the negative gradient. As can be seen, the angle approaches 90° *i.e.* the inner product of the gradient and the normal to the plane approaches zero as the function of the iteration number. In this case and with the chosen step size we obtain the accuracy of $|\epsilon| \leq 0.01$ within 10 iterations.

The projections of the negative gradient, the actual constraint and the linearized constraint onto the plane (a_2, a_4) after 30 iterations are shown in Fig. 1. The unconstrained minimum and one error contour are also shown in Fig. 1.

Table 1: The angle ψ (in degrees) between the active constraint plane and the negative gradient, and the termination criterion $|\epsilon|$ as functions of the iteration.

Iteration	1	10	30
ψ	86.781	89.576	89.995
$\|\epsilon\|$	0.056	0.010	$1 \cdot 10^{-4}$

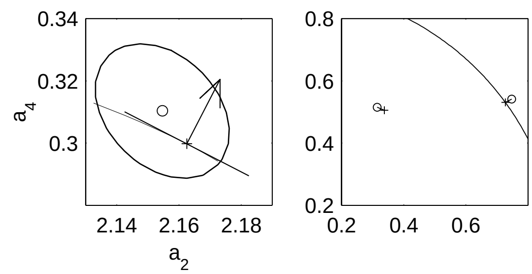

Figure 1: Left: the projections of the negative gradient (arrow), the actual constraint (weak line) and the linearized constraint (heavy line) onto the plane (a_2, a_4) after 30 iterations, the unconstrained minimum (small circle) and one error contour. Right: a section of the circle with radius ρ in the complex plane and the roots that correspond to the solutions of the unconstrained ('∘') and constrained ('+') problems.

The constrained minimum corresponds to the situation in which the negative gradient points directly outward from the the feasible region, that is, the gradient is normal to the tangent plane of the constraints.

The coefficients and the roots corresponding to the local minimum are

$$\begin{aligned} a &= (-2.1294, 2.1624, -1.0854, 0.2998) \\ \lambda_{1,2} &= 0.900\exp(\pm 0.201\pi\mathtt{i}) \\ \lambda_{3,4} &= 0.608\exp(\pm 0.313\pi\mathtt{i})\ . \end{aligned}$$

The coefficient estimates satisfy clearly the hyperstability constraints. The roots that correspond to the unconstrained and constrained problems are shown in Fig. 1. We observe that the addition of the constraints moves the unstable root to the edge of feasible region and furthermore, the other root is also adjusted.

4.2. The time-varying case

In the time-varying case we applied the proposed method to a time-varying EEG sample. We constructed a TVAR(6) model for the sample with gaussian basis functions ($M = 6$) and applied this model for frozen time spectrum estimation [3]. The unconstrained and corresponding stability constrained spectrum estimate is shown (with $\mu = .2$ and $\rho = 0.95$) in Fig. 2. The unconstrained spectrum estimate is clearly

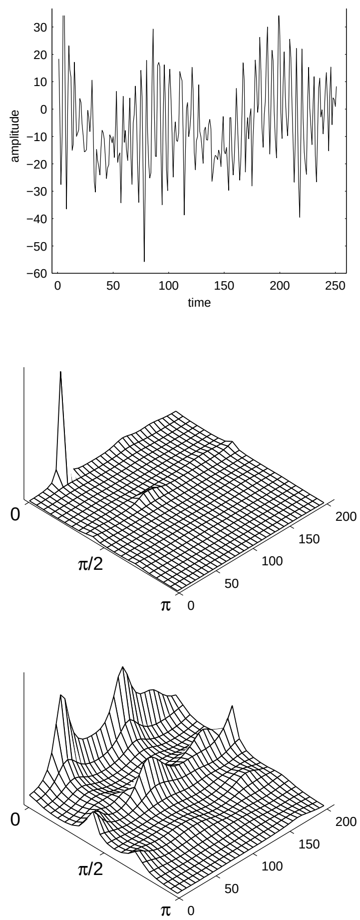

Figure 2: Up) EEG sample, Middle) The time varying amplitude spectrum estimate of the unstabilized TVARLS scheme and Down) The corresponding estimate of the stabilized TVARLS scheme.

useless for further applications but the estimate obtained with the hyperstability constrained estimation scheme can be applicable for clinical purposes.

5. DISCUSSION

We have proposed an algorithm for the hyperstabilization of autoregressive models. While there are several methods that guarantee classical stability, it seems that there have been only ad hoc methods for the hyperstabilization of the models, such as radial pole adjustment in the stationary case, which can not be applied to the time-varying case.

Although we have discussed only AR modeling and specifically the (nonwindowed, forward prediction) least squares estimation of the coefficients, the method is clearly applicable to other AR parameter estimation methods, such as the Yule-Walker method. The method is also relatively easily modified for filter design problems when hyperstability is required. For adaptive algorithms hyperstability can be implemented e.g. as in [10, 2], but block algorithms seem not to have been proposed earlier.

However, the most relevant field of application of the proposed method is the estimation of TVARLS models, or equivalently the time-varying linear prediction coding (TV-LPC) modeling. In these cases the hyperstability can often be a practical prerequisite for the applicability of the TVARLS methods.

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