NUMERICAL STABILITY ISSUES OF THE CONVENTIONAL RECURSIVE LEAST SQUARES ALGORITHM

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ABSTRACT

The continuous use of adaptive algorithms is strongly dependent on their behavior in finite-precision environments. We study the nonlinear round-off error accumulation system of the conventional RLS algorithm and we derive bounds for the relative precision of the computations and the accumulated round-off error, which guarantee the numerical stability of the finite-precision implementation of the algorithm. The bounds depend on the conditioning of the problem and the exponential forgetting factor. Simulations agree with our theoretical results.

1. INTRODUCTION

A very important "real-life" problem, inherent in the continuous use of adaptive algorithms, is their behavior in finite precision environments. This problem contains the following subproblems: round-off error generation, round-off error propagation, and round-off error accumulation.

For the conventional RLS algorithms the round-off error propagation is the best studied of the three aforementioned subproblems [1], [2]. Such studies typically examine the linearized round-off error propagation system and derive its exponentially stability; this in turn, implies *local* exponential stability of the *nonlinear* round-off error accumulation system. However, no study exists, to our knowledge, that provides an indication as to "how small" the accumulated error should be so that the influence of the nonlinear terms does not destroy the stability properties of the overall system.

An examination of the nonlinear round-off error accumulation system of the conventional RLS algorithm appeared in [3], where a scenario for explosive divergence was developed. Explosive divergence is the occurence of "sudden big" errors in the estimated weight vector, due to finite precision effects. This phenomenon is linked to the loss of the positive definiteness of the finite precision inverse covariance matrix, and the negative value of a theoretically positive variable. However, the approach is mostly qualitative and numerical stability cannot be guaranteed. This is our main subject. We study the stability properties of the *nonlinear* round-off error accumulation system of the conventional RLS algorithm and we derive

- An upper bound for the relative precision of the computations, in terms of the condition number of the problem and the forgetting factor, which guarantees that the nonlinear round-off error accumulation system remains BIBO stable.
- An upper bound for the accumulated round-off error.

2. RECURSIVE LEAST-SQUARES ALGORITHMS

For the standard least squares problem, one is given a sequence of M–dimensional input vectors, ϕ_t, plus a reference sequence, $u_t, t = 1, \ldots, k$, and is asked to compute a parameter vector, θ_k, such that

$$\theta_k = \arg\min \sum_{t=1}^{k} \lambda^{k-t}(u_t - \theta^t \phi_t)^2, \tag{1}$$

where λ is the forgetting factor. The CLS algorithm is given by the following recursions:

$$r_k^e = \lambda + \phi_k^t P_{k-1}\phi_k \tag{2}$$

$$\theta_k = \theta_{k-1} + \frac{P_{k-1}\phi_k}{r_k^e}(u_k - \theta_{k-1}^t \phi_k) \tag{3}$$

$$P_k = \frac{1}{\lambda}\left(P_{k-1} - \frac{P_{k-1}\phi_k\phi_k^t P_{k-1}}{r_k^e}\right) \tag{4}$$

Denoting $\widetilde{P}_k$ the finite-precision version of P_k, we may express the finite-precision time update of P_k as

$$\widetilde{P}_k = \frac{1}{\lambda}\left(\widetilde{P}_{k-1} - \frac{\widetilde{P}_{k-1}\phi_k\phi_k^t \widetilde{P}_{k-1}}{\widetilde{r}_k^e}\right) + \epsilon\widetilde{P}_k, \tag{5}$$

where the term $\epsilon\widetilde{P}_k$ denotes the local round-off error in the computation of P_k.

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3. THE ROUND-OFF ERROR ACCUMULATION SYSTEM

Let Δx be the accumulated round-off error in x. Then

$$\Delta P_k = \widetilde{P}_k - P_k, \tag{6}$$
$$\Delta r_k^e = \widetilde{r}_k^e - r_k^e = \phi_k^t \Delta P_{k-1}\phi_k. \tag{7}$$

If $\left|\frac{\Delta r_k^e}{r_k^e}\right| < 1$ we can expand the second term of (5) as

$$\frac{\widetilde{P}_{k-1}\phi_k\phi_k^t\widetilde{P}_{k-1}}{\widetilde{r}_k^e} = \frac{P_{k-1}\phi_k\phi_k^t P_{k-1} + 2P_{k-1}\phi_k\phi_k^t\Delta P_{k-1} + \Delta P_{k-1}\phi_k\phi_k^t\Delta P_{k-1}}{r_k^e}\times$$

$$\underbrace{\Big(1 - \underbrace{\frac{\Delta r_k^e}{r_k^e} + \underbrace{\left(\frac{\Delta r_k^e}{r_k^e}\right)^2 - \cdots}_{t_2(k,\Delta P_{k-1})}}_{t_1(k,\Delta P_{k-1})}\Big)}_{t_0(k,\Delta P_{k-1})}, \tag{8}$$

where we have used the fact that ΔP_{k-1} is symmetric, which is necessary for the convergence of the algorithm in finite precision [4], and trivially imposed in a finite precision implementation. Thus

$$\Delta P_k = \frac{1}{\lambda}\left(\Delta P_{k-1} + \frac{P_{k-1}\phi_k\phi_k^t P_{k-1}}{r_k^e}\frac{\Delta r_k^e}{r_k^e} - \frac{2P_{k-1}\phi_k\phi_k^t\Delta P_{k-1}}{r_k^e}\right)$$
$$-\frac{1}{\lambda}\left(\frac{P_{k-1}\phi_k\phi_k^t P_{k-1}}{r_k^e}t_2(k,\Delta P_{k-1}) + \right.$$
$$+\frac{2P_{k-1}\phi_k\phi_k^t\Delta P_{k-1}}{r_k^e}t_1(k,\Delta P_{k-1})$$
$$\left. + \frac{\Delta P_{k-1}\phi_k\phi_k^t\Delta P_{k-1}}{r_k^e}t_0(k,\Delta P_{k-1})\right) + \epsilon\widetilde{P}_k. \tag{9}$$

The study of (9) is of primordial importance for the "real-life" finite-precision implementation of the CLS algorithm. However, it seems that the existence of the higher-order terms (inside the second set of parentheses) has been a major obstacle towards this purpose. In the sequel we study the nonlinear difference equation (9) and we derive sufficient conditions for its BIBO stability, which implies numerical stability of the CLS algorithm.

3.1. Assumptions

1. The regressor vector ϕ_t is persistently exciting, that is, $\exists\, a, b$ such that $0 < a < b < \infty$ and

$$a\,I \le \sum_{t=1}^{k}\lambda^{k-t}\phi_t\phi_t^t \le b\,I, \quad \text{for all } k > k_0. \tag{10}$$

Thus, there exist constants $\mathcal{R}$ and $\mathcal{P}$ such that

$$||R_k|| \le \mathcal{R}, \quad \text{and} \quad ||P_k|| \le \mathcal{P}, \quad \text{for all } k, \tag{11}$$

where $||\cdot||$ denotes the 1-norm. The validity of the bounds in (11) for $k < k_0$ can be guaranteed by a "soft" start. If we assume that

$$||\phi_t|| \le \Phi, \quad \text{for all } t, \tag{12}$$

then $\mathcal{R}$ can be expressed as

$$\mathcal{R} \equiv \frac{\Phi^2}{1-\lambda}, \tag{13}$$

and an upper bound for the condition number of the estimated data covariance matrix is

$$\mathcal{K} \stackrel{\text{def}}{=} \mathcal{R}\mathcal{P} \equiv \frac{\mathcal{P}\Phi^2}{1-\lambda}. \tag{14}$$

2. The round-off error $\epsilon\widetilde{P}_k$ is bounded as

$$\left|\left|\epsilon\widetilde{P}_k\right|\right| \le \mathcal{E}\,\epsilon, \quad \text{for all } k, \tag{15}$$

where ϵ denotes the relative precision of the computations and $\mathcal{E}$ is a bounded constant. In [5] we provide a detailed description of the round-off error introduced at each iteration of the CLS algorithm and we estimate the value of $\mathcal{E}$.

4. STABILITY ANALYSIS OF THE ROUND-OFF ERROR ACCUMULATION SYSTEM

If $f(k,\Delta P_{k-1})$ denote the higher-order terms of (9), then

$$\Delta P_k = \lambda P_k R_{k-1}\,\Delta P_{k-1}\,R_{k-1}P_k + f(k,\Delta P_{k-1}) + \epsilon\widetilde{P}_k. \tag{16}$$

Looking at the linearized system, we see that

$$\Delta P_k = \lambda^{k-i}P_k R_i\,\Delta P_i\,R_i P_k, \tag{17}$$

which gives

$$||\Delta P_k|| \le \lambda^{k-i}\mathcal{K}^2\,||\Delta P_i||. \tag{18}$$

That is, the linearized round-off error propagation system is exponentially stable with base of decay λ [1]. This implies that the nonlinear round-off error propagation system is locally exponentially stable. However, no study exists, to our knowledge, that provides an indication as to "how small" ΔP_k should be, so that the influence of the nonlinear and the additive terms does not destroy the stability properties of the nonlinear round-off error accumulation system. This is our main task in the sequel.

At first, we derive the solution of (16) as

$$\Delta P_k = \sum_{i=1}^{k} \lambda^{k-i} P_k R_i \left(f(i, \Delta P_{i-1}) + \epsilon P_i\right) R_i P_k, \quad (19)$$

with $f(1, \Delta P_0) = 0$. Then, we assume that $||\Delta P_i|| \le r$, $i = 1, \ldots, k-1$, and we provide an upper bound for $||\Delta P_k||$.

Theorem 1: If $||\Delta P_i|| \le r$, for $i = 1, \ldots, k-1$, and $r < \frac{\lambda}{\Phi^2}$, then

$$||\Delta P_k|| \le \frac{\overbrace{\Phi^2 \left(\mathcal{K} + \mathcal{P}\Phi^2\right)^2}^{\mathcal{A}_1} r^2}{\lambda(1-\lambda)(\lambda - \Phi^2 r)} + \frac{\mathcal{K}^2\, \mathcal{E}\, \epsilon}{1-\lambda}. \quad (20)$$

The proof can be found in [5].

If we can find an r, independent of k, in the range $0 < r < \frac{\lambda}{\Phi^2}$, such that the right-hand side of (20) is less than or equal to r, i.e.,

$$||\Delta P_k|| \le \frac{\mathcal{A}_1 r^2}{\lambda(1-\lambda)(\lambda - \Phi^2 r)} + \frac{\mathcal{K}^2 \mathcal{E} \epsilon}{1-\lambda} \le r, \quad (21)$$

then by induction $||\Delta P_k|| \le r$, for all k, meaning that the round-off error accumulation system is BIBO stable.

Setting $\epsilon = 0$ in (21) we derive the bound:

$$r \le r_0 \stackrel{\text{def}}{=} \frac{\lambda^2(1-\lambda)}{\mathcal{A}_1 + \underbrace{\lambda(1-\lambda)\Phi^2}_{\mathcal{A}_2}}. \quad (22)$$

For $r \in (0,\, r_0]$, we guarantee that $||\Delta P_k|| \le r$, for all k, if

$$\epsilon \le \frac{1-\lambda}{\mathcal{K}^2 \mathcal{E}} \underbrace{\left(r - \frac{\mathcal{A}_1 r^2}{\lambda(1-\lambda)(\lambda - \Phi^2 r)}\right)}_{F_0(r)}. \quad (23)$$

In order to maximize the relative precision ϵ (that is, minimize the wordlength) that guarantees BIBO stability of the round-off error accumulation system, we have to maximize the function $F_0(r)$ in the interval $(0, r_0]$. The extremal points of $F_0(r)$ are the solutions of the second order equation

$$\Phi^2(\mathcal{A}_1 + \mathcal{A}_2) r^2 - 2\lambda(\mathcal{A}_1 + \mathcal{A}_2) r + \frac{\lambda^2}{\Phi^2}\mathcal{A}_2 = 0. \quad (24)$$

Maximization is achieved at

$$\rho_1^0 = \frac{\lambda}{\Phi^2}\left(1 - \frac{\sqrt{\mathcal{A}_1^2 + \mathcal{A}_1\mathcal{A}_2}}{\mathcal{A}_1 + \mathcal{A}_2}\right). \quad (25)$$

Upper bounds for the relative precision and the accumulated round-off given the condition number $\mathcal{K}$, the forgetting factor λ, and the size of the round-off error $\mathcal{E}$, are

$$\epsilon \le \epsilon_0 \stackrel{\text{def}}{=} \frac{1-\lambda}{\mathcal{K}^2 \mathcal{E}}\, F_0(\rho_1^0), \quad ||\Delta P_k|| \le \rho_1^0,\ \forall k. \quad (26)$$

The bounds (26) seem to be *conservative*, mainly because in the proof of Theorem 1 [5], we have used the condition number $\mathcal{K}$ as an upper bound for $||P_k R_i||$. The bounds so obtained apply in the general nonstationary case. Sharper bounds, however, can be obtained if the input data are stationary, as we now pursue.

4.1. The Stationary Case

When the input sequence, ϕ_t, is stationary, then in steady-state and for λ very close to 1, [6], [3]

$$P_k R_{k-1} \approx I, \ \text{ for large } k. \quad (27)$$

This approximation affords the derivation of bounds which are more realistic than the ones derived in the previous section. The round-off error accumulation system is

$$\Delta P_k = \lambda \Delta P_{k-1} + f(k, \Delta P_{k-1}) + \epsilon \widetilde{P}_k \quad (28)$$

The next theorem [5], provides a bound for $||\Delta P_k||$.

Theorem 2: If ϕ_t is a stationary sequence, $||\Delta P_i|| \le r$ for $i = 1, \ldots, k-1$, and $r < \frac{\lambda}{\Phi^2}$, then

$$||\Delta P_k|| \le \frac{\overbrace{\Phi^2 \left((1-\lambda)\mathcal{P}\Phi^2 + 3 - 2\lambda\right)}^{\alpha_1} r^2}{\lambda(1-\lambda)(\lambda - \Phi^2 r)} + \frac{\mathcal{E}\epsilon}{1-\lambda} \ (\le r) \quad (29)$$

Setting in (29) $\epsilon = 0$ gives an upper bound for r, as

$$r \le r_1 \stackrel{\text{def}}{=} \frac{\lambda^2(1-\lambda)}{\alpha_1 + \mathcal{A}_2}. \quad (30)$$

For each $r \in (0, r_1]$, if

$$\epsilon \le \frac{1-\lambda}{\mathcal{E}} \underbrace{\left(r - \frac{\alpha_1 r^2}{\lambda(1-\lambda)(\lambda - \Phi^2 r)}\right)}_{F_1(r)}, \quad (31)$$

then $||\Delta P_k|| \le r$ for all k. We derive a bound for ϵ, by maximizing $F_1(r)$ for $r \in (0, r_1]$, as

$$\epsilon \le \epsilon_1 \stackrel{\text{def}}{=} \frac{1-\lambda}{\mathcal{E}} F_1(\rho_1^1), \quad (32)$$

where

$$\rho_1^1 = \frac{\lambda}{\Phi^2}\left(1 - \frac{\sqrt{\alpha_1^2 + \alpha_1\mathcal{A}_2}}{\alpha_1 + \mathcal{A}_2}\right). \quad (33)$$

The corresponding bound for the accumulated round-off accumulated is

$$||\Delta P_k|| \le \rho_1^1,\ \forall k. \quad (34)$$

The bound (32) is much less conservative than (26), mainly because the condition number $\mathcal{K}$ has been replaced by unity. It can be shown that [5]

$$\epsilon \le \epsilon_1 < \frac{1-\lambda}{\mathcal{K}}, \quad (35)$$

which establishes the relation between the conditioning of the problem and the numerical stability of the CLS algorithm, which has been claimed in the general context of adaptive algorithms [7]. We also observe that the relative precision is proportional to $1 - \lambda$, which means that for λ very close to 1 the round-off error accumulation is more significant, as to be expected.

5. SIMULATIONS

We derived upper bounds on the relative precision, ϵ_0 (resp. ϵ_1) that guarantee that the accumulated round-off is bounded by ρ_1^0 (resp. ρ_1^1). This gives sufficient conditions for the BIBO stability of the round-off error accumulation system and provides an upper bound for the accuracy of the computations.

In order to check out our theoretical results we generate input data using an AR model with poles $.85$, $.7 \pm .4j$, $-.4 \pm .6j$. We run in double precision floating point arithmetic the CLS algorithm with order $M = 5$ and $\lambda = .99$ and we derive estimates for $\mathcal{P}$ and Φ. Then, we use formulas (32), (34) and an estimate of $\mathcal{E}$ [5], to derive upper bounds for the relative precision and the accumulated round-off error, as $\epsilon_1 = 1.3961 \times 10^{-6}$ and $\rho_0^1 = 4.0668 \times 10^{-4}$, respectively. We run the CLS algorithm in floating point with 20 bit precision as indicated by the value of ϵ_1. In Figure 1 we plot the accumulated round-off error, $||P_k - \widetilde{P}_k||$, which satisfies the theoretically predicted bound (we used the double precision variables as the reference variables). We have repeated this experiment with many different types of data and for millions of iterations and always the accumulated error satisfied the predicted by our theory bound.

6. CONCLUSIONS

We have considered the problem of finite-precision implementation of the CLS algorithm, and in particular we have computed upper bounds for the relative precision that guarantee the BIBO stability of the accumulated round-off error. These bounds depend on the conditioning of the problem, $\mathcal{K}$, and the forgetting factor, λ. Most previous studies have considered a linearized (and hence, oversimplified) round-off error accumulation system and thus cannot establish bounds on the relative precision that guarantee numerical stability of the algorithm. Our approach resembles a numerical analysis one, which means that the derivation of the bounds is based on the application of the triangle and submultiplicative norm inequalities. This fact makes the derived bounds somewhat conservative, especially in the general nonstationary case.

7. REFERENCES

[1] S. Ljung and L. Ljung, "Error Propagation Properties of Recursive Least-squares Adaptation Algorithms," *Automatica*, Vol. 21, No. 2, pp. 157-167, 1985.

[2] M. H. Verhaegen, "Round-off Error Propagation in Four Generally-applicable, Recursive, Least-squares Estimation Schemes," *Automatica*, vol. 25, No. 3, pp. 437-444, 1989.

[3] G. Bottomley and S. T. Alexander, "A Novel Approach for Stabilizing Recursive Least Squares Filters," *IEEE Trans. Signal Processing*, vol. 39, no. 8, August 1991.

[4] D. T. M. Slock, "Backward consistency concept and round-off error propagation dynamics in resursive least-squares algorithms," *Optical Engineering*, Vol. 31, No. 6, pp. 1153-1169, June 1992.

[5] A. P. Liavas and P. A. Regalia, "On the numerical stability and accuracy of the conventional recursive least-squares algorithm,", submitted for publication to the *IEEE Trans. Signal Processing*, May 1996.

[6] E. Eleftheriou and D. Falconer, "Tracking properties and Steady-State Performance of RLS Adaptive Filter Algorithms," *IEEE Trans. Acoust., Speech, Signal Proc.*, vol. ASSP-34, no. 5, pp 1097-1109, October 1986.

[7] J. M. Cioffi, "Limited precision effects in adaptive filtering," *IEEE Trans. Circuits Systems.*, vol CAS-34, pp 821-833, July 1987.

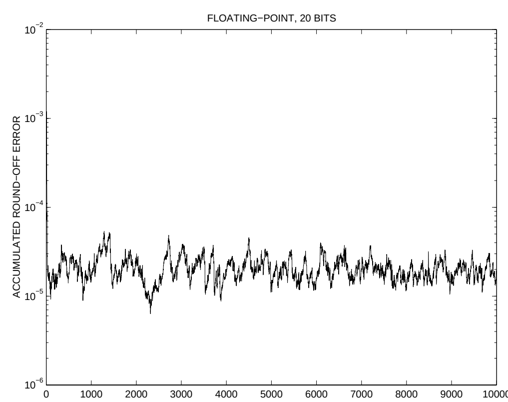

Figure 1. Accumulated round-off error, floating point arithmetic: 20 bit precision.